Linear-Threshold Modeling of Brain Network Dynamics

Erfan Nozari

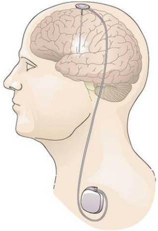
University of California, San Diego
Department of Mechanical and Aerospace Engineering
Department of Cognitive Science
http://carmenere.ucsd.edu/erfan

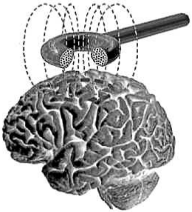
Joint work with: Prof. Jorge Cortés

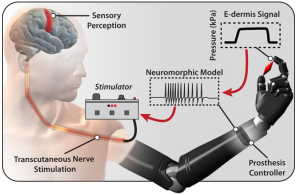
2019 American Control Conference
July 9, 2019

Overview

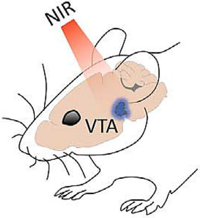
- Brain as a networked dynamical system
- *New:* rapid advancements in neuro-technologies
- Critical applications in
  - ✓ Deep brain stimulation (DBS)
  - ✓ Transcranial magnetic stimulation (TMS)
  - ✓ Brain-machine/computer interfaces (BMI/BCI)
  - ✓ Optogenetics
  - ✓ ⋮







[Osborn et al, Sci Robot, 2018]



[Chen et al, Science, 2018]

Outline

1. Derivation

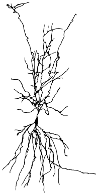
2. Analysis

Outline

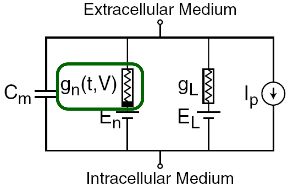
1. Derivation

2. Analysis

- Conductance-based (a.k.a. Hudgkin-Huxley) models:

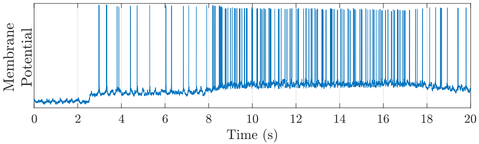


Neuron ≡ RC Circuit



- Input = current, output = voltage
- **Nonlinear (active) & time-varying resistors** ⇒ excitable behavior (spiking)

$$\textbf{output} \simeq \sum_{t_s} \delta(t - t_s)$$

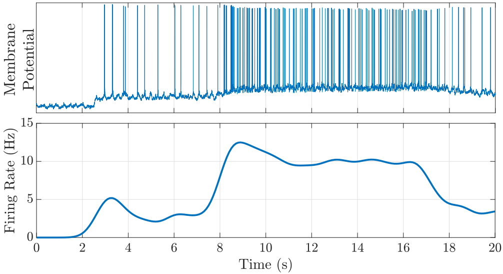


Data from [Henze et al, CRCNS, 2009]

[Image Att: Behrang Amini, Wikimedia.org]

Mean-Field Approximation: Rate Dynamics

- Often, it seems that information mostly encoded in **firing rate** (#spikes/s)



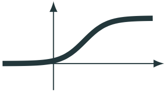
- $x_i(t) =$ **firing rate** of neuron i

- Simplifying **assumptions**:

1. Poisson spiking

2a. For constant input $I_{\text{in},i}$

$$x_i = \sigma(I_{\text{in},i})$$



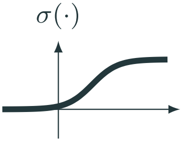
2b. For time-varying input $I_{\text{in},i}(t)$

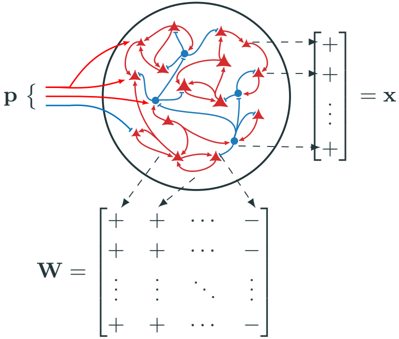
$$\tau \dot{x}_i(t) = -x_i(t) + \sigma(I_{\text{in},i}(t))$$

3. Slowly varying inputs $I_{\text{in},i}(t)$ ($\gg \tau$)

- Node = population of neurons
- State = average firing rate
- Network **dynamics** (mean-field approximation):

$$\tau \dot{\mathbf{x}}(t) = -\mathbf{x}(t) + \sigma\big(\mathbf{W}\mathbf{x}(t) + \mathbf{p}(t)\big)$$





Approximating the Sigmoidal Nonlinearity

Two popular approximations:

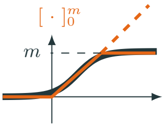
✓ **Kuramoto:** Cubic approximation in $\mathbf{x}_i$, linearization in $\{W_{ij}\}$, change to polar coordinates

$$\dot{\theta}_i = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i)$$

$\rightarrow$ For weakly-coupled oscillators, explicit phase dynamics, $\frac{n}{2}$ states, smooth

✓ **Linear-Threshold:** Piecewise-linearization of $\sigma(\cdot)$

$$\tau_i \dot{x}_i = -x_i + \left[\sum_j W_{ij} x_j + p_i\right]_0^{m_i}$$



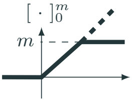
$\rightarrow$ **For arbitrary dynamics**, implicit phase and amplitude (oscillations), switched-affine

Outline

1. Derivation

2. Analysis

$$\tau_i \dot{x}_i = -x_i + \Big[\underbrace{\sum_j W_{ij} x_j + p_i}_{I_{\mathrm{in},i}} \Big]_0^{m_i}$$



- ✓ Solution exist in the classical sense (C^1) and is unique
- ✓ State space:

$$[\mathbf{0}, \mathbf{m}] = [0, m_1] \times [0, m_2] \times \cdots \times [0, m_n]$$

- ✓ Dynamics of each node i can be in 3 modes $\Rightarrow 3^n$ switching regions

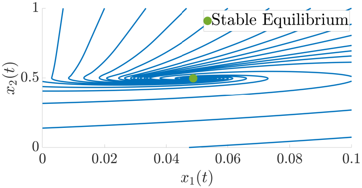
$$\begin{cases} \tau_i \dot{x}_i = -x_i & \text{if} \quad I_{\mathrm{in},i} \leq 0 \\ \tau_i \dot{x}_i = -x_i + I_{\mathrm{in},i} & \text{if} \quad 0 \leq I_{\mathrm{in},i} \leq m_i \\ \tau_i \dot{x}_i = -x_i + m_i & \text{if} \quad m_i \leq I_{\mathrm{in},i} \end{cases}$$

- ✓ Switched-affine representation:

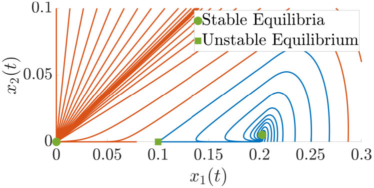
$$\tau \dot{\mathbf{x}} = (-\mathbf{I} + \mathbf{\Sigma}^{\ell}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{W}) \mathbf{x} + \mathbf{\Sigma}^{\ell}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{p} + \mathbf{\Sigma}^{\mathrm{s}}_{\boldsymbol{\sigma}(\mathbf{x})} \mathbf{m}, \qquad \boldsymbol{\sigma}(\mathbf{x}) \in \{0, \ell, \mathrm{s}\}^n$$

✓ Wide range of complex behavior, including

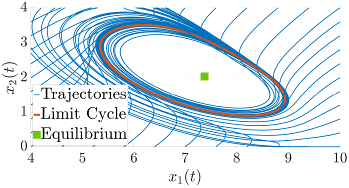
1. Monostability



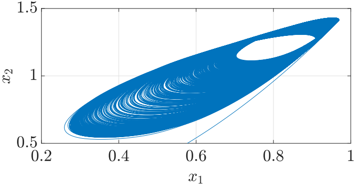
2. Multistability



3. Limit cycles



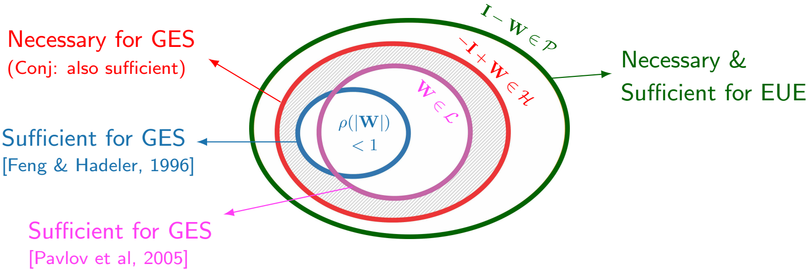
4. Chaos



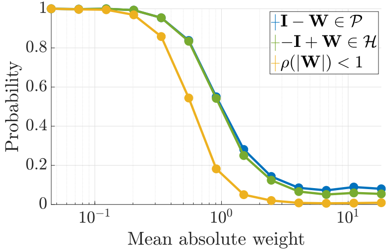
Some definitions:

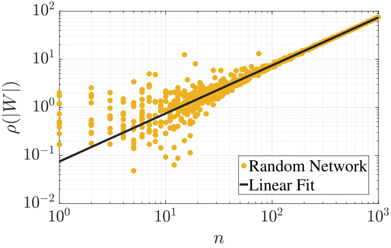
- $\mathbf{W} \in \mathcal{H}$ if all its principal submatrices are Hurwitz
- $\mathbf{W} \in \mathcal{P}$ if all its principal minors are positive
- $\mathbf{W} \in \mathcal{L}$ if there exists $\mathbf{P} = \mathbf{P}^T > \mathbf{0}$ such that for all $\boldsymbol{\sigma} \in \{0,1\}^n$

$$(-\mathbf{I} + \mathbf{W}^T \mathrm{diag}(\boldsymbol{\sigma}))\mathbf{P} + \mathbf{P}(-\mathbf{I} + \mathrm{diag}(\boldsymbol{\sigma})\mathbf{W}) < \mathbf{0}$$



- The **stronger** or **larger** a network, the **more unstable** it becomes





? Brain networks are large and become stronger with learning (without losing stability!)

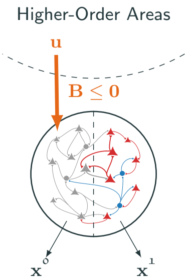
$\Rightarrow$ Need for **stabilization mechanisms**:

- via structure $\mathbf{W} \rightarrow$ homeostasis (re-normalizing rows of $\mathbf{W}$)
- via input $\mathbf{p}(t) \rightarrow$ **?**

Selective Stabilization via Inhibitory Control

- Input decomposition: $\mathbf{p}(t) = \mathbf{B}\mathbf{u}(t) + \tilde{\mathbf{p}}$
- Stabilization can/should be **selective**

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{0} \\ \mathbf{x}^{1} \end{bmatrix} \begin{array}{l} \longrightarrow \text{to be stabilized} \\ \longrightarrow \text{arbitrary (active)} \end{array}$$



Theorem: Inhibitory Stabilization

Assume

$$\mathbf{u}(t) \equiv \bar{\mathbf{u}} \qquad \text{or} \qquad \mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$$

If $\dim(\mathbf{u}) \geq \dim(\mathbf{x}^{0})$, there exists $\mathbf{u}(t)$ such that

$$\mathbf{x}(t) \xrightarrow{\text{GES}} \mathbf{x}^{*} = (\mathbf{0}, \mathbf{x}^{*1})$$

if and only if the $\mathbf{x}^{1}$ sub-dynamics is internally GES

$\Rightarrow$ The **stability of $\mathbf{x}^{1}$** is the sole determiner of the **stabilizability of $\mathbf{x}$**

- Layer dynamics:

$$\tau_i \dot{\mathbf{x}}_i(t) = -\mathbf{x}_i(t) + \left[\mathbf{W}_{i,i}\mathbf{x}_i(t) + \mathbf{p}_i(t)\right]_{\mathbf{0}}^{\mathbf{m}}$$

1. Selective activity/stabilization:

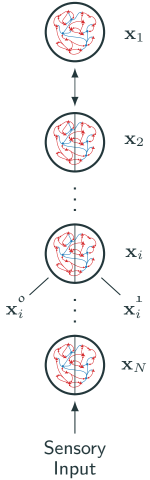
$$\mathbf{x}_i = \begin{bmatrix}\mathbf{x}_i^0 \\ \mathbf{x}_i^1\end{bmatrix} \begin{matrix}\longrightarrow \text{to be stabilized} \\ \longrightarrow \text{arbitrary (active)}\end{matrix} \quad , \quad \mathbf{W}_{i,j} = \begin{bmatrix}\mathbf{W}_{i,j}^{00} & \mathbf{W}_{i,j}^{01} \\ \mathbf{W}_{i,j}^{10} & \mathbf{W}_{i,j}^{11}\end{bmatrix}$$

2. Chain topology (information processing pathways):

$$\mathbf{p}_i(t) = \mathbf{B}_i\mathbf{u}_i(t) + \mathbf{W}_{i,i-1}\mathbf{x}_{i-1}(t) + \mathbf{W}_{i,i+1}\mathbf{x}_{i+1}(t) + \mathbf{c}_i$$

3. Timescale separation:

$$\tau_1 > \tau_2 > \cdots > \tau_i > \cdots > \tau_N$$



Theorem: Hierarchical Stabilization & Tracking

Assume $\dim(\mathbf{u}_i) \geq \dim(\overset{\circ}{\mathbf{x}}_i)$ for all i. There exists

$$\mathbf{u}_i(t) = \mathbf{K}_i \mathbf{x}_i(t) + \bar{\mathbf{u}}_i(t), \qquad \forall i$$

such that

$$\forall i \quad \begin{cases} \overset{\circ}{\mathbf{x}}_i(t) \to \mathbf{0} & \text{(Inhibitory Stabilization)} \\ \overset{1}{\mathbf{x}}_i(t) \to \mathbf{x}_i^{*1}(\mathbf{W}^{11}_{i,i-1}\overset{1}{\mathbf{x}}_{i-1}(t) + \overset{1}{\mathbf{c}}_i) & \text{(Tracking)} \end{cases}$$

as

$$\frac{\tau_i}{\tau_{i-1}} \to 0, \qquad \forall i$$

if

$$\tau_i \dot{\overset{1}{\mathbf{x}}}_i(t) = -\overset{1}{\mathbf{x}}_i(t) + [\mathbf{W}^{11}_{i,i}\overset{1}{\mathbf{x}}_i(t) + \mathbf{W}^{11}_{i,i+1}\mathbf{x}^{*1}_{i+1}(\mathbf{W}^{11}_{i+1,i}\overset{1}{\mathbf{x}}_i(t) + \overset{1}{\mathbf{c}}_{i+1}) + \overset{1}{\mathbf{c}}_i]^+$$

is GES for all $\overset{1}{\mathbf{c}}_{i+1}$ and $\overset{1}{\mathbf{c}}_i$

Extensions to Hierarchical Structures – cont'd

1. **Equilibrium maps**

> **Lemma: Piecewise-Affine Equilibrium Maps**
>
> The equilibrium of layer i is given by
>
> $$\mathbf{x}_i^*(\mathbf{x}_{i-1}) = \mathbf{F}_{i,\lambda}\mathbf{x}_{i-1} + \mathbf{f}_{i,\lambda}, \ \forall \mathbf{x}_{i-1} \in \Psi_{i,\lambda}, \ \lambda \in \Lambda_i$$
>
> where $\{\mathbf{F}_{i,\lambda}, \mathbf{f}_{i,\lambda}, \Psi_{i,\lambda}, \Lambda_i\}$ have recursive expressions

2. **Multi-layer GES**

> **Theorem: Global Exponential Stability (GES)**
>
> Let $\bar{\mathbf{F}}_i \triangleq \max_{\lambda \in \Lambda_i} |\mathbf{F}_{i,\lambda}|$. If
>
> $$\rho(|\mathbf{W}_{i,i}| + |\mathbf{W}_{i,i+1}|\bar{\mathbf{F}}_{i+1}|\mathbf{W}_{i+1,i}|) < 1$$
>
> then $\mathbf{x}_i^{1}(t)$ is GES for all $\mathbf{c}_{i+1}^{1}$ and $\mathbf{c}_i^{1}$.

3. **Time-scale separation:** $\frac{\tau_i}{\tau_{i-1}} \leq \frac{1}{1.5}$ is often enough in practice

1. Data: [Rodgers & DeWeese, Neuron, 2014]
2. Defining nodes (clustering neurons)
3. Computing $\mathbf{x}(t)$
4. Defining edges (brain physiology)
5. Finding edge weights:

$$\min_{\theta} \quad d(\mathbf{x}_{\text{data}}, \mathbf{x}_{\text{model}})$$

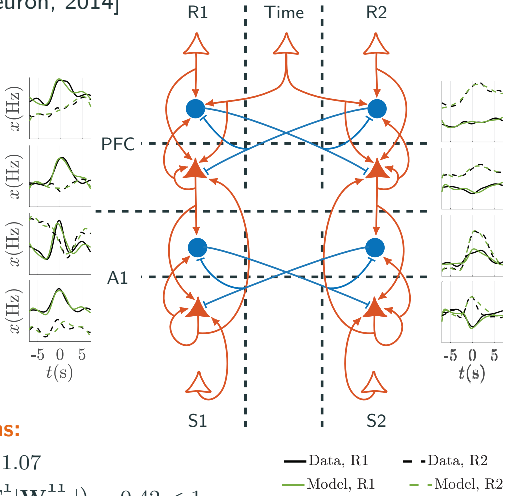
$$\theta = [w_{i,j}, b_{i,j}, c_i, \tau_i, x_i(0)]_{i,j}$$

6. **Verifying theoretical conditions:**

✓ $\tau_1 = 4.70 \gg \tau_2 = 2.33 \gg \tau_3 = 1.07$

✓ Under R1: $\rho\big(|\mathbf{W}^{11}_{2,2}| + |\mathbf{W}^{11}_{2,3}|\bar{F}^1_3|\mathbf{W}^{11}_{3,2}|\big) = 0.42 < 1$

✓ Under R2: $\rho\big(|\mathbf{W}^{11}_{2,2}| + |\mathbf{W}^{11}_{2,3}|\bar{F}^1_3|\mathbf{W}^{11}_{3,2}|\big) = 0.13 < 1$



Beyond Equilibrium Attractors: Neural Oscillations

- **Attractor dynamics:** dynamics that settle to a stable pattern (manifold)

👍 Facilitate analysis

👎 Miss transients (unless $\mathbf{x}(0)$ close to attractor)

- Common forms:
  1. Equilibrium attractors
     - Isolated equilibria, as above

$$\forall i \quad \begin{cases} \mathbf{x}_i^{\mathtt{o}}(t) \to \mathbf{0} & \text{(Inhibitory Stabilization)} \\ \mathbf{x}_i^{\mathtt{1}}(t) \to \mathbf{x}_i^{*\,\mathtt{1}}(\mathbf{W}_{i,i-1}^{\mathtt{11}}\mathbf{x}_{i-1}^{\mathtt{1}}(t) + \mathbf{c}_i^{\mathtt{1}}) & \text{(Tracking)} \end{cases}$$

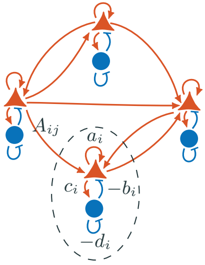
     - Continuum of equilibria (line, ring, plane, ...)
  2. **Oscillatory attractors**
     - **Limit cycles (regular)**
     - **Chaotic oscillations (irregular/noisy-like)**

Structural Characterization of Oscillations

- Network of Wilson-Cowan oscillators

$$\tau_i \dot{x}_{i,1} = -x_{i,1} + \left[a_i x_{i,1} - b_i x_{i,2} + p_{i,1} + \sum\nolimits_j A_{ij} x_{j,1}\right]_0^{m_{i,1}}$$
$$\tau_i \dot{x}_{i,2} = -x_{i,2} + \left[c_i x_{i,1} - d_i x_{i,2} + p_{i,2}\right]_0^{m_{i,2}}$$

- Lack of stable equilibria (LoSE) as proxy for oscillations



Oscillator i

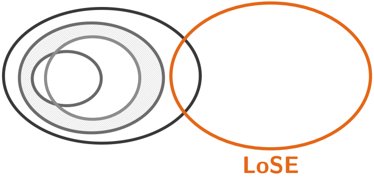
Theorem: Lack of Stable Equilibria

For each oscillator i, **LoSE** *iff*

$$d_i + 2 < a_i$$
$$(a_i - 1)(d_i + 1) < b_i c_i$$
$$(a_i - 1) m_{i,1} < b_i m_{i,2}$$
$$\underline{p}_{i,\ell} < p_{i,\ell} < \bar{p}_{i,\ell}, \ \ell = 1, 2$$

and, if so, for the full network, **LoSE** *iff*

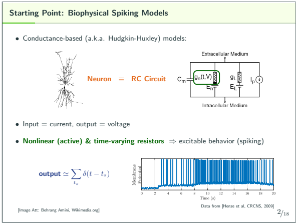
$$\exists i : \quad p_{i,1} + \sum\nolimits_j A_{ij} m_{j,1} < \bar{p}_{i,1}$$



Summary

In this talk:

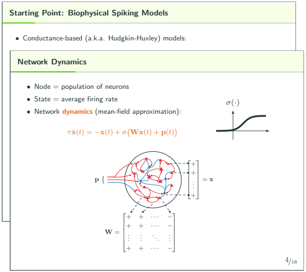
✓ Biophysical spiking models



Summary

In this talk:

- ✓ Biophysical spiking models
- ✓ Mean-field approximation $\Rightarrow$ rate models



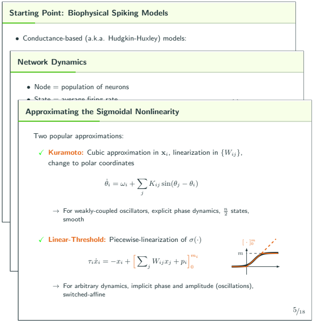
Summary

In this talk:

✓ Biophysical spiking models

✓ Mean-field approximation $\Rightarrow$ rate models

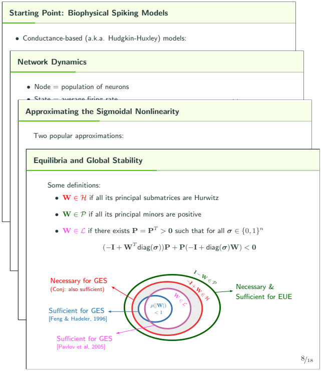
✓ Linear-threshold approximation



Summary

In this talk:

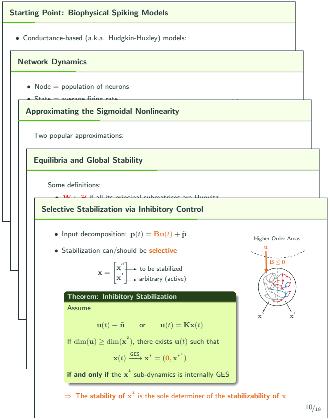
- ✓ Biophysical spiking models
- ✓ Mean-field approximation ⇒ rate models
- ✓ Linear-threshold approximation
- ✓ Stability analysis



Summary

In this talk:

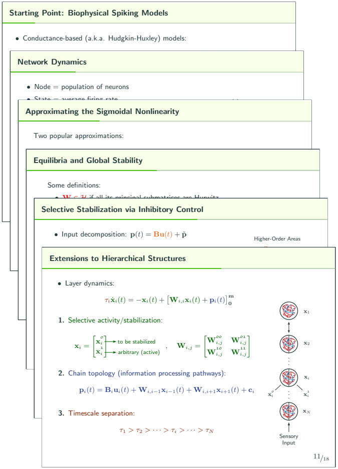
- ✓ Biophysical spiking models
- ✓ Mean-field approximation ⇒ rate models
- ✓ Linear-threshold approximation
- ✓ Stability analysis
- ✓ Stabilization via inhibition



Summary

In this talk:

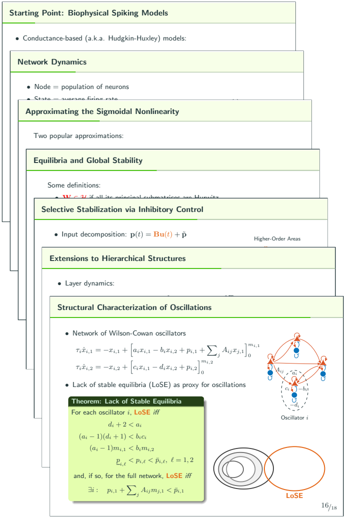
- ✓ Biophysical spiking models
- ✓ Mean-field approximation ⇒ rate models
- ✓ Linear-threshold approximation
- ✓ Stability analysis
- ✓ Stabilization via inhibition
- ✓ Hierarchical structures



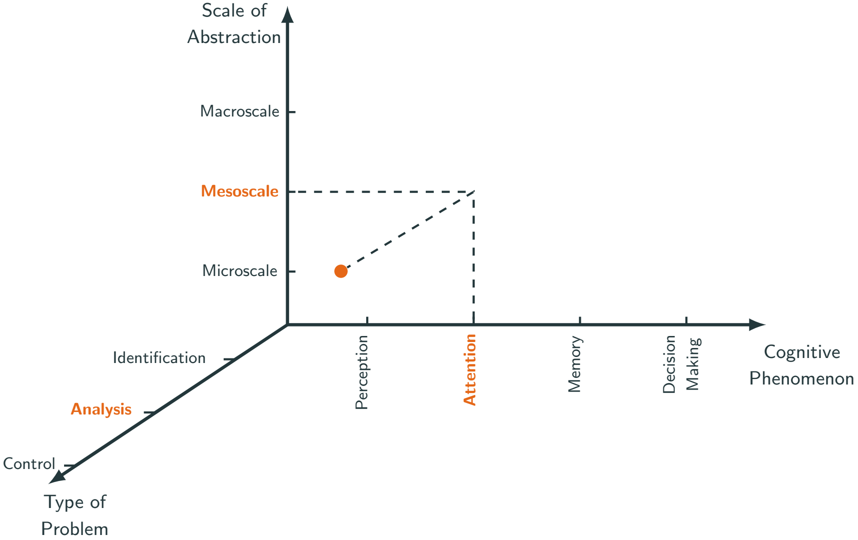
Summary

In this talk:

- ✓ Biophysical spiking models
- ✓ Mean-field approximation ⇒ rate models
- ✓ Linear-threshold approximation
- ✓ Stability analysis
- ✓ Stabilization via inhibition
- ✓ Hierarchical structures
- ✓ Oscillatory attractors



Research Directions



Thank You!

Questions and Comments?

Extended results available at:

- Hierarchical Selective Recruitment in Linear-Threshold Brain Networks
  Part I: Intra-Layer Dynamics and Selective Inhibition
  E. Nozari, J. Cortés, `https://arxiv.org/abs/1809.01674`
- Hierarchical Selective Recruitment in Linear-Threshold Brain Networks
  Part II: Inter-Layer Dynamics and Top-Down Recruitment
  E. Nozari, J. Cortés, `https://arxiv.org/abs/1809.02493`
- Oscillations and Coupling in Interconnections of Two-Dimensional Brain Networks
  E. Nozari, J. Cortés, ACC'19