Control of Networks

Algorithms, Fundamental Limitations, Impossibility Results

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- This is the subject of this presentation. Designed to be self-contained (no knowledge of control necessary...)

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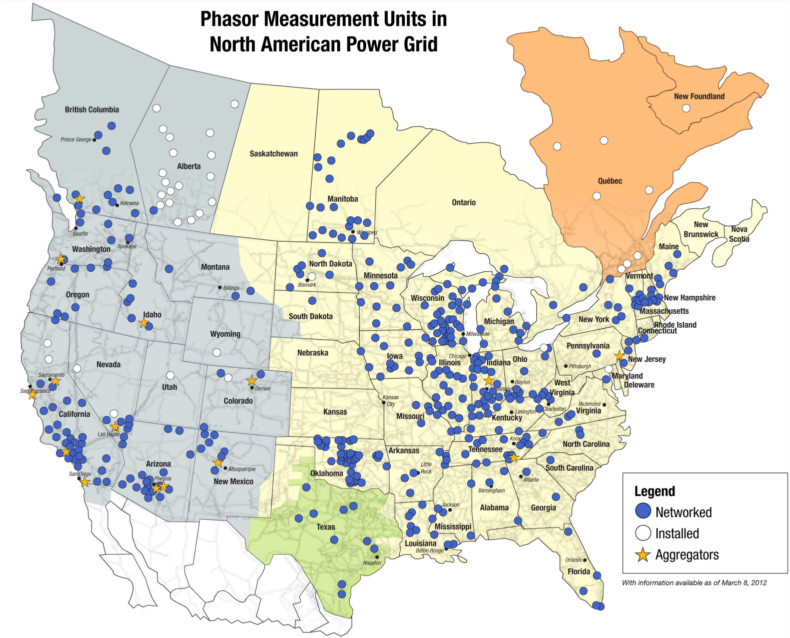
- This is part of the North American Synchronophasor Initiative. Goal is described as 100% coverage of important transmission lines.

PMU Placement as of 2015



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  3. Energy constrained control: controllability with a bound on control energy (for example, to move from the origin to a random point on the unit sphere).

Energy considerations

- Given
$$\dot{x} = Ax + Bu,$$
let $E(x_i \to x_f, T)$ be the energy it takes to drive the system from x_i to x_f:
$$E(x_i \to x_f, T) = \inf\{\int_0^T ||u(t)||_2^2 \, dt \mid u \text{ drives the system from } x_i \text{ to } x_f\}$$

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- In every real world scenario, use of arbitrarily large inputs in unphysical.
- Very easy to write down reasonable-looking real 10×10 systems where the energy is of the magnitude 10^{30} or more.
- Want to measure "difficulty of controllability" through just one number. Standard choice:

$$\mathcal{E}(T) = \frac{1}{S_1} \int_{||z||_2=1} E(0 \to z, T) \, dz,$$

  where S_1 is the surface area of the unit sphere.

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ABSTRACT

This paper presents the formulation of a class of optimization problems dealing with selecting, at each instant of time, one measurement provided by one out of many sensors. Each measurement has an associated measurement cost. The basic problem is then to select an optimal measurement policy, during a specified observation time interval, so that a weighted combination of "prediction accuracy" and accumulated "observation cost" is optimized. The current analysis is limited to the class of linear stochastic dynamic systems and measurement subsystems. The problem of selecting the optimal measurement strategy can be transformed into a deterministic optimal control problem. An iterative digital computer algorithm is suggested for obtaining numerical results. It is shown that the optimal measurement policy and the associated "matched" Kalman-type filter can be precomputed, i.e., specified before the measurements actually occur.

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- I will consider this in discrete time. Will dispense with formal problem statement.

A very partial literature review

- [Simon and Mitter, *Information and Computation*, 1968]. Considers minimizing the number of driver nodes.
- [Athans, *Automatica*, 1972]. An optimal control approach to time-varying actuator scheduling.
- Many works on structural controllability of networks in the 1980-early 1990s by Shields, Pearson, Glover, Willems, Siljak, Commault, Dion...
- Much work (recent and old) on sensor placement for observation of PDEs.
- [Liu, Slotine, Barabasi, *Nature*, 2011] explain to minimize the number of driver nodes generically for controllability.
- [Muller, Schuppert, *Nature*, 2011] argues most practical results involve affecting only a small number of key variables.
- [O., *IEEE Trans. on Control of Network Systems*, 2014] How to achieve controllability while minimizing the number of variables affected?
- Generalizations by O., Pappas, Jadbabaie, Bushnell, Poovendran, Lygeros, Cortes, Belabbas, Pasqualetti, Pequito, and their students to reachability and minimizing control energy.
- [Jadbabaie, O., Siami, *IEEE Trans. on Automatic Control* submission]

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- As observed in follow-up papers, as a consequence reachability and energy-efficient control are also NP-hard problems.

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- ...in many cases, this is good enough! For example, if the linear system is controllable from $O(1)$ entries, this finds $O(\ln n)$ entries.

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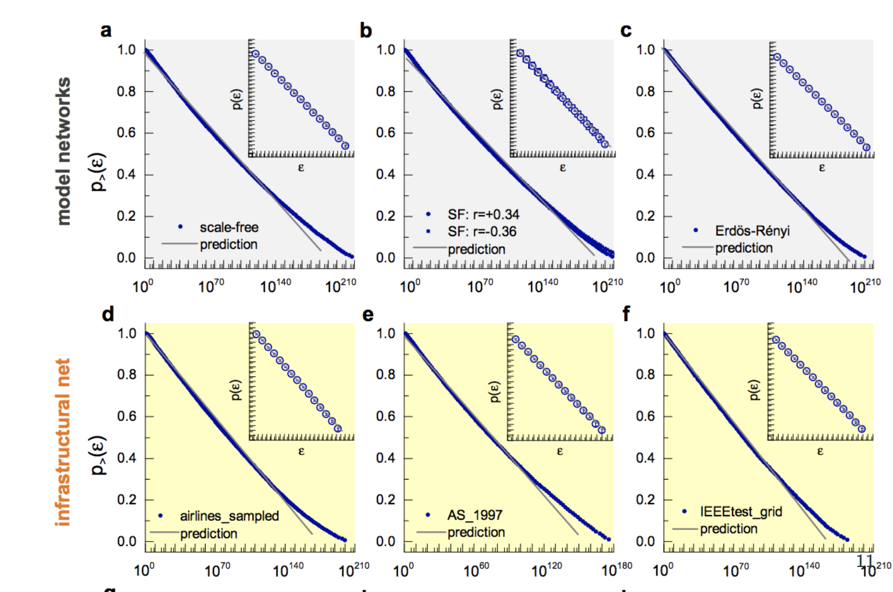
- The $O(\log n)$ approximability is a general result about greedy supermodular optimization.

Unfortunately, energy grows exponentially with these methods



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- If true, the same guarantees would effortlessly carry over. Unfortunately, I constructed a counterexample in [O., *IEEE Transactions on Control of Network Systems*, 2018].

Non-supermodularity of control energy

- Somewhat of a counterintuitive phenomenon. Counterexample:

$$A = \begin{pmatrix} -182 & 0 & -565 & 0 & -11 & -736 \\ 0 & -1075 & 831 & -276 & -1752 & -612 \\ -565 & 831 & -2435 & 214 & 1321 & -1853 \\ 0 & -276 & 214 & -73 & -453 & -158 \\ -11 & -1752 & 1321 & -453 & -2864 & -1045 \\ -736 & -612 & -1853 & -158 & -1045 & -3371 \end{pmatrix}$$

$$\begin{aligned} \mathcal{E}_{I=\{1,2,3\}}(\infty) - \mathcal{E}_{I=\{1,2,3,4\}}(\infty) &\approx 2.5 \cdot 10^4 \\ \mathcal{E}_{I=\{1,2,3,5\}}(\infty) - \mathcal{E}_{I=\{1,2,3,4,5\}}(\infty) &\approx 2.52 \cdot 10^4 \end{aligned}$$

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- Approximation with energy constraints is currently an open problem.

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- In fact, we show a bit more....

Third main result

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- Punchline: minimal reachability is almost exponentially harder than minimal controllability, which was approximable to a factor of $O(\log n)$.

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- In fact, lets limit ourselves to choosing an average of d actuators per step.

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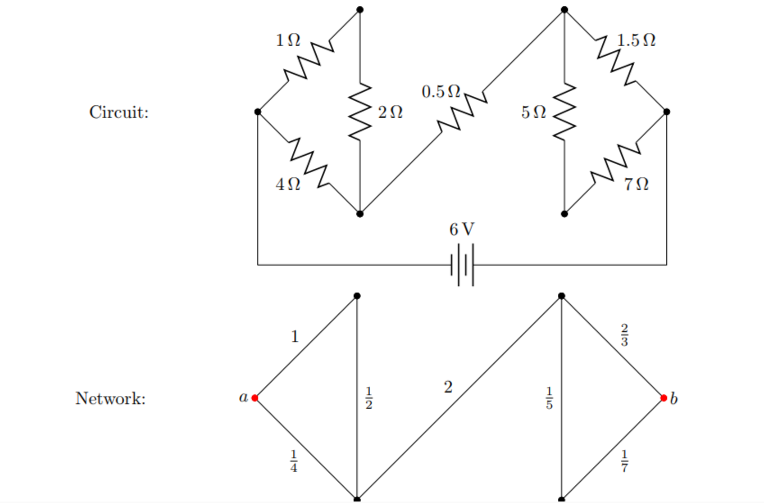
- A graph can be encoded as a Laplacian matrix (defined by putting w_{ij}, the weight between the i'th and j'th node into the (i, j)'th entry of the matrix.
- Given a graph with a Laplacian L, we are asking for a sparse subgraph (say with $O(n)$ edges) with a Laplacian L_s such that $L \approx L_s$.

Resistance in a graph

A key observation is that you can define the resistance between any two nodes:



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- How can we use generalize this to our setting?
- Key idea: sample actuator i with probability proportional to $P_i^T W(T)^{-1} P_i$ where P_i is the i'th column of $W(T) = \int_0^T e^{tA} B B^T e^{tA^T}\, dt$. This is independent across time.

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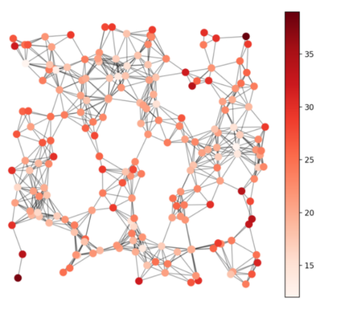
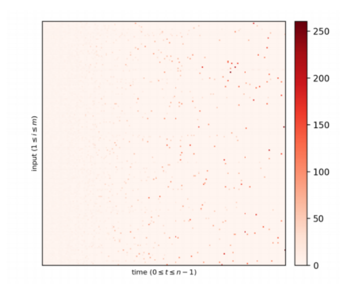
- **Theorem:** [Jadbabaie, O., Siami, *IEEE Trans. on Automatic Control* submission, 2019] If $T \geq n$ and d is at least a constant multiple of $\log n/\epsilon^2$ then with high probability the control energy of this scheme is at most $1 + \epsilon$ times the best possible (here, best possible means actuating every variable).

Performance on a 250 agent network

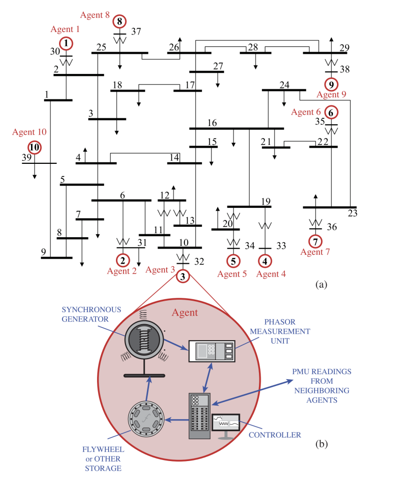


The system matrix A is the Laplacian of this undirected graph, with colors corresponding to weights on edges. The matrix is semistable.



A sparse actuator schedule. A dot corresponds to a used actuator at a given time, and the color corresponds to size of the corresponding input.

The IEEE 39-bus system



This is a way to represent a complete graph with equations $m\ddot{\theta}_i + d_i\dot{\theta}_i = -\sum_j k_{ij}(\theta_i - \theta_j) + u_i$ coupling the nodes. This matrix is semistable.

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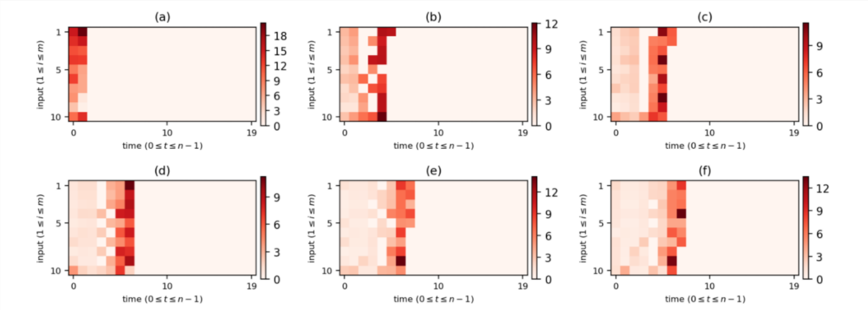

Colors represent intensity of the actuator use. In contrast to the previous example, this schedule seems to be "front-loaded."

Conclusion

- Key takeaways:
  1. Exist optimal algorithm for minimal controllability.
  2. Supermodularity is a key property. It's lack makes things difficult.
  3. Minimal reachability is, surprisingly, close to being unsolvable.
  4. Effective control with time-varying actuators has been almost solved.
- Main challenge: find a class of systems for which incorporating energy constraints and desired reachable directions can be done.
- Satisfactory answers could have a transformative impact not only in electricity distribution systems and many other areas.