Controllability, Fragility, and Tradeoffs in Complex Networks

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Analysis and Control of Complex Networks

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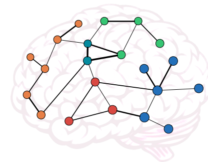
Complex networks



Brain network

Power network

Social network

Analyze, predict, and control **dynamics** over **large networks**

- Asymptotic properties, scaling laws, tradeoffs
- Controllability, resilience, fragility, synchronization, spectrum ...
  ... (natural/social/technological systems)
- Algebraic graph theory, systems theory, sparse matrix analysis

In this talk

1. Controllability bounds for complex networks
   - Many complex networks are difficult to control
   - Some (anisotropic) networks are easy to control
2. Design networks with prescribed controllability profiles
   - Networks with diagonal controllability Gramian
3. Controllability tradeoffs in complex networks
   - Controllability can be improved only at the expenses of robustness

Problem formulation

System description

$$x(t+1) = Ax(t) + Bu(t) \qquad (\text{or } \dot{x}(t) = Ax(t) + Bu(t))$$

- The matrix A is sparse (interaction graph)
- $B = [e_{i_1}\ e_{i_2}\ \cdots\ e_{i_m}]$ where $e_i = \underbrace{[0\ \ 0\ \ \cdots\ \ 0\ \ 1\ \ 0\ \ \cdots\ \ 0]^\mathsf{T}}_{i\text{-th component equals } 1}$

Controllability definitions

$$\text{Controllability Gramian:} \qquad \mathcal{W}_T = \sum_{t=0}^{T-1} A^t B B^\mathsf{T} A^t$$

Small $\lambda_{\min}(\mathcal{W}_T)$ $\Leftrightarrow$ **Small controllability degree**

Large $\lambda_{\min}(\mathcal{W}_T)$ $\Leftrightarrow$ **Large controllability degree**

Few nodes cannot control symmetric networks

Upper bound on controllability degree

$$\lambda_{\min}(\mathcal{W}_T) \leq \frac{\mu^{2\lceil n(\mu)/m \rceil - 2}}{1-\mu^2}$$

- $0 < \mu < 1$
- $n(\mu) = |\{\lambda \,:\, \lambda \in \text{spec}(A), |\lambda| \leq \mu\}|$

Typically, $n(\mu)$ grows linearly with the network cardinality:

1. For fixed number of control nodes, the controllability degree **decreases exponentially** with the network cardinality
2. For fixed controllability degree, the number of control nodes **must grow linearly** with the network cardinality
3. Counterintuitive result w.r.t. structural controllability

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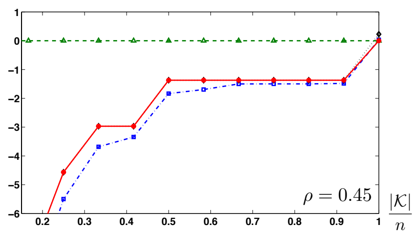
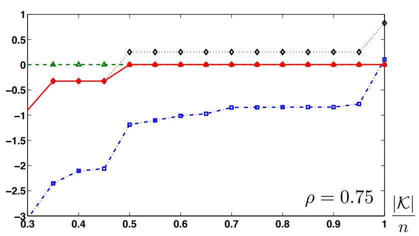
Network matrix Schur stable $\Rightarrow$ $n(\mu) = n$

Upper bound with $\mu = \lambda_{\max}$

$$\lambda_{\min}(\mathcal{W}_T) \leq \frac{\lambda_{\max}^{2\lceil n/m \rceil - 2}(A)}{1-\lambda_{\max}^2(A)}$$

$$A = \frac{\rho}{3}\begin{bmatrix} 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 0 & 1 & 1 \end{bmatrix}$$





The smaller the spectral radius, the tighter the bound

Can few nodes control asymmetric networks?

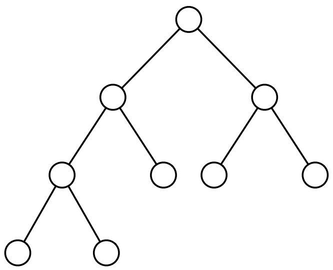
Upper bound on controllability degree

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \text{cond}^2(V)\frac{\mu^{2\lceil n(\mu)/m \rceil - 2}}{1-\mu^2}$$

- V is an eigenvector matrix of A
- $\text{cond}(V) = \sigma_{\max}(V)/\sigma_{\min}(V)$ is the condition number of V

1. If $\text{cond}(V)$ remains bounded with the network dimension ...
   ... then the network remains difficult to control
2. If $\text{cond}(V)$ grows with the network dimension ...
   ... then the network may be easy to control

Controllability of acyclic networks #1

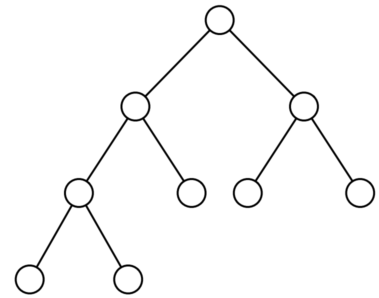


- $A = [a_{ij}]$
- $0 < a_{\min} \leq a_{ij} \leq a_{\max} < \infty$
- $V =$ eigenvector matrix
- How does $\text{cond}(V)$ scale with n?

Facts:

- A acyclic $\Rightarrow$ there exists $D = \text{diag}(d_{ij})$ s.t. $\tilde{A} = DAD^{-1}$ symmetric
- $\tilde{A}$ symmetric $\Rightarrow$ there exists $\tilde{A}W = W\Lambda$ with $\text{cond}(W) = 1$
- $\tilde{A}W = W\Lambda \;\Rightarrow\; A(D^{-1}W) = (D^{-1}W)\Lambda \;\Rightarrow\; V = D^{-1}W$
- $\text{cond}(V) = \|V\|\|V^{-1}\| \leq \|D\|\|D^{-1}\|\|W\|\|W^{-1}\| \leq \text{cond}(D)\underbrace{\text{cond}(W)}_{=1}$

Controllability of acyclic networks #2

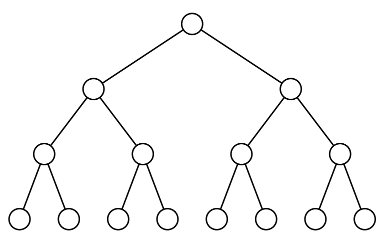


- $0 < a_{\min} \leq a_{ij} \leq a_{\max} < \infty$
- $D = \text{diag}(d_{ij})$ s.t. DAD^{-1} symmetric
- $DAD^{-1}W = W\Lambda$, $\text{cond}(W) = 1$
- $\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \text{cond}^2(D)\frac{\mu^{2\lceil n(\mu)/m\rceil - 2}}{1-\mu^2}$
- $\text{cond}(D)$ depends on network diameter
- $\text{cond}(D) \leq \left(\frac{a_{\max}}{a_{\min}}\right)^{\text{diameter}}$

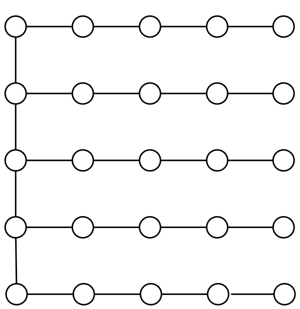
Controllability of acyclic networks

Acyclic networks with sublinear diameter ($o(n)$) are difficult to control
(by constant number of control nodes)

Controllability of acyclic networks #3



Diameter $\in O(\log n)$



Diameter $\in O(\sqrt{n})$

So far...

Networks that are difficult to control

- Symmetric networks
- Acyclic networks with $o(n)$ diameter
- Networks with constant condition number

Are there networks that are easy to control?

Outline

1. Controllability bounds for complex networks
   - Many complex networks are difficult to control
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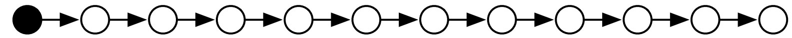
   - Controllability can be improved only at the expenses of robustness

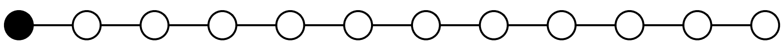
Certain asymmetric networks are easy to control



$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}$$

1. The network is stable with identity controllability matrix
2. The controllability degree is independent of the network cardinality

Controllability of Toeplitz line networks #1



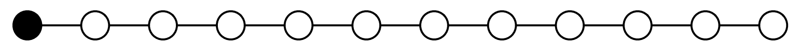
$$A = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \cdots & 0 \\ 0 & c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a & b \\ 0 & \cdots & \cdots & c & a \end{bmatrix}$$

Controllability of line networks

A toeplitz line network is **easy** to control if one of the following holds:

- $\frac{a(b+c)}{4bc} \leq 1$ and $1 < (b-c)^2(1 - \frac{a^2}{4bc})$
- $\frac{a(b+c)}{4bc} > 1$ and $1 \leq c + b - a$

Controllability of Toeplitz line networks #2



Controllability of line networks

A toeplitz line network is **difficult** to control if one of the following holds:

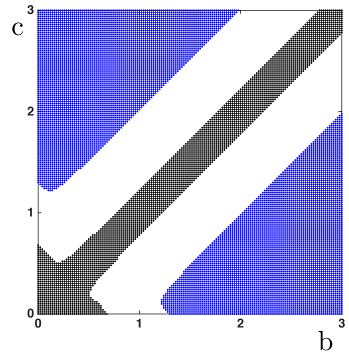
- $a + b + c < 1$
- $ab < c < \frac{b}{a}$ and $a \geq \sqrt{bc}$
- $ba^{n_a} < c < \frac{b}{a^{n_a}}$ and $a < \sqrt{bc}$ with $n_a = \frac{2}{\pi} \arccos\left(\frac{-a}{\sqrt{bc}}\right) - 1$

- Easy to control $\Rightarrow$ Energy is bounded independent of cardinality
- Difficult to control $\Rightarrow$ Energy increases exponentially with cardinality

(with constant number of control nodes)

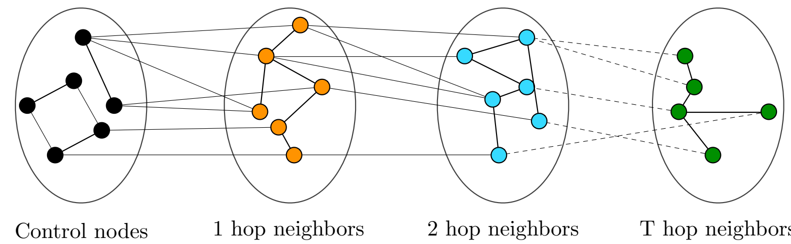
Controllability of Toeplitz line networks #3

$$A = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \cdots & 0 \\ 0 & c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a & b \\ 0 & \cdots & \cdots & c & a \end{bmatrix}$$



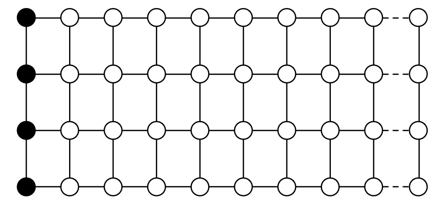
1. Given a, if c **is sufficiently larger than** b, then line networks are controlled with finite energy, independent of the network dimension
2. **Spatial instability** is exploited to have high controllability degree (large non-normality degree)
3. The width of the uncontrollable region depends on the parameter a

Extension to general networks



Easy to control line-like networks:

- forward matrices of full rank
- k-neighbors $\leq$ $(k-1)$-neighbors
- condition on matrices: $\|C_i^\dagger\|_2^{\max}(1+\|D_i\|_2^{\max}+\|B_i\|_2^{\max}) < 1$



Then...

Networks that are difficult to control

- Symmetric networks
- Acyclic networks with $o(n)$ diameter
- Networks with constant condition number

Networks that are easy to control

- Sufficiently anisotropic networks with triangular controllability matrix
  - Toeplitz line networks
  - block tridiagonal networks
- Sufficiently anisotropic cycle networks

Are there *interesting* networks that are easy to control?

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Networks with prescribed controllability Gramian #1

Can we design networks with prescribed Gramian (i.e., contr. degree)?

- Focus on the case where $\mathcal{W}$ is diagonal (restrictive yet insightful)
- Continuous-time network dynamics

Sign-skew-symmetric network

The network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with input nodes $\mathcal{S}$ is sign-skew-symmetric if

- $a_{ij} a_{ji} < 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise;
- $a_{ii} < 0$ if $i \in \mathcal{S}$, and $a_{ii} = 0$ otherwise.

Control impact along a path

For a sign-skew-symmetric network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with input nodes $\mathcal{S}$,

- the control impact along the path $(i_1, i_2, \dots i_p)$, with $i_i \in \mathcal{S}$ is

$$\beta_{i_1,\dots,i_p} = \frac{1}{|a_{i_1 i_1}|} \left|\frac{a_{i_2 i_1}}{a_{i_1 i_2}}\right| \left|\frac{a_{i_3 i_2}}{a_{i_2 i_3}}\right| \cdots \left|\frac{a_{i_p i_{p-1}}}{a_{i_{p-1} i_p}}\right|$$

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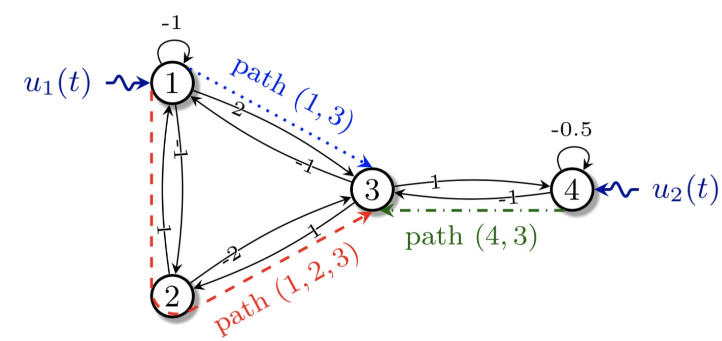
Networks with diagonal controllability Gramian #2



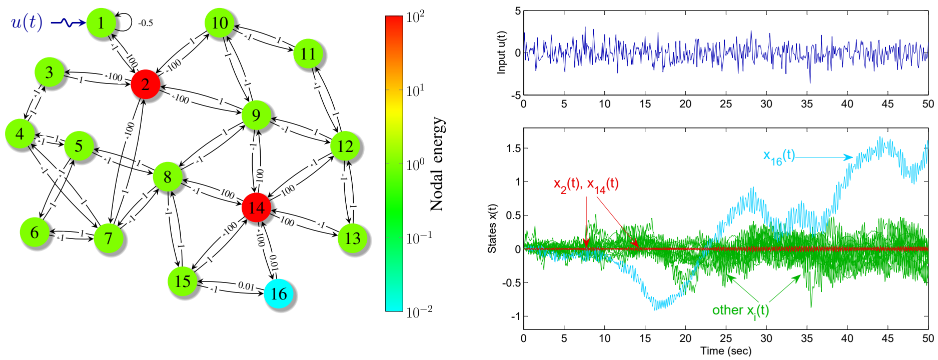
Uniformly input-connected network

The network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with input nodes $\mathcal{S}$ is uniformly input-connected if

- it is sign-skew-symmetric, and
- for every node i, all control impacts to i are equal to $\beta_i \in \mathbb{R}_{>0}$.

Networks with diagonal controllability Gramian #3

Stable + uniformly input-connected $\iff$ $\mathcal{W} = \text{diag}(\beta_1, \dots, \beta_n)$



Select weights to assign control impacts $\Rightarrow$ Control energy by design

Implications of networks with diagonal Gramian

1. Large class or continuous-time networks is easy to control
   - That is, control energy may not increase with network cardinality
2. Energy to control a node may be independent of graphical distance
   - That is, distant nodes may require little energy to change their state
3. We can create networks with desired nodal energies
   - That is, we can secure selected nodes from exogenous disturbances

Is there a price to pay for easy controllability?

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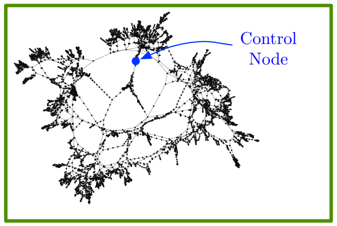
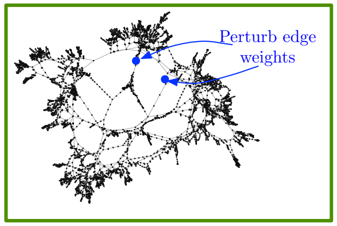
Two (independent?) properties of dynamical networks



For a dynamic network, **controllability** is the ability to drive the state to arbitrary configurations through external controls

For a dynamic network, **fragility** refers to the ability to maintain a stable behavior in the face of small edge perturbations



A definition of network fragility

Network dynamics

$$\dot{x}/x^+ = Ax + Bu$$

Fragility definition

Stability radius: $r(A) = \min\{\|\Delta\| \; : \; A + \Delta \text{ is not Hurwitz stable}\}$

Small $r(A)$ $\Leftrightarrow$ **Large fragility degree**

Large $r(A)$ $\Leftrightarrow$ **Small fragility degree**

- Small $r(A) \Rightarrow$ easy to destabilize network
- Perturbation Δ is complex and non-structured
- Equivalently, $r(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(i\omega I - A) = \frac{1}{\max_{\omega \in \mathbb{R}} \|(i\omega I - A)^{-1}\|}$

Why are natural and technological networks fragile? #1

Empirical evidence and theories suggest that large networks are fragile ...
... (ecological [Nat.'72], power [Chaos'07], neural [NeuralComp.'14])

Controllability vs fragility in networks

$$\lambda_{\min}(\mathcal{W}) \leq \bar{\lambda}(\mathcal{W}) \leq \frac{n_c}{n}\left(1 + \frac{4\|A - A^\mathsf{T}\|}{3\pi}\frac{1}{r(A)}\right)\frac{1}{r(A)}$$

- n = cardinality of the network
- n_c = number of control nodes
- $\bar{\lambda}(\mathcal{W})$ = average eigenvalue of Gramian $\mathcal{W}_\infty$

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Why are natural and technological networks fragile? #2

Controllability vs fragility in networks

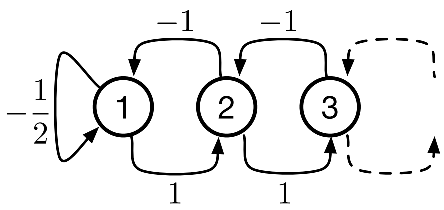
$$\lambda_{\min}(\mathcal{W}) \leq \bar{\lambda}(\mathcal{W}) \leq \frac{n_c}{n}\left(1 + \frac{4\|A - A^\mathsf{T}\|}{3\pi}\frac{1}{r(A)}\right)\frac{1}{r(A)}$$

1. few control nodes $n_c \Rightarrow$ large control energy
2. large stability radius $r(A) \Rightarrow$ small Gramian eigenvalues ...
   ... (robust networks cannot be easy to control)
3. if cardinality n increases $\Rightarrow$ controllability and/or robustness decrease
   ... (assuming terms in bracket are sublinear in n)

For symmetric networks:

$$\lambda_{\min}(\mathcal{W}) \leq \bar{\lambda}(\mathcal{W}) \leq \frac{n_c}{n}\frac{1}{r(A)}$$

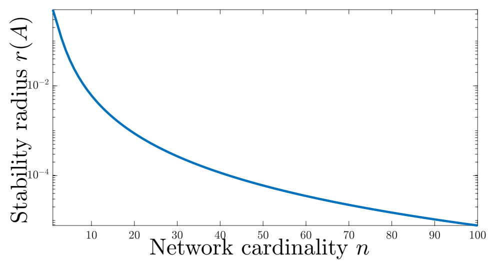
An example: easily controllable but fragile network



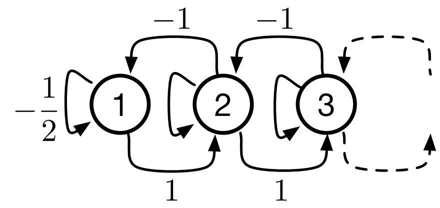
$$A = \begin{bmatrix} -1/2 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

- control node $= \{1\}$
- $\mathcal{W} = I \Rightarrow$ easy to control

As n increases, contr. degree is fixed, and fragility increases



An example: robust but poorly controllable network



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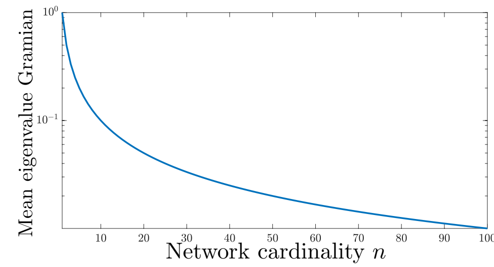
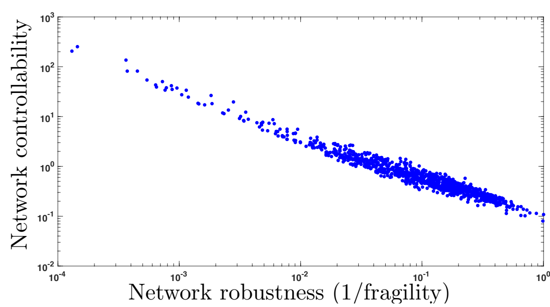
- control node $= \{1\}$
- A is normal; $r(A) = 0.5$

As n increases, fragility degree is fixed, and contr. decreases



A fundamental tradeoff between controllability and fragility

controllability $\leq$ (network parameters) $\cdot$ **fragility**



- Controllability implies fragility in complex networks
- Fundamental tradeoffs, valid for many (linear) network models
- Predator-prey dynamics, neural systems, traffic networks ...

Summary

In this talk

1. Metrics for network controllability and bounds
2. Design of networks that are easy to control
3. Controllability vs fragility tradeoff

Research directions

1. Conjecture: networks with $o(n)$ diameter are difficult to control
2. Criteria for networks that are easy to control
3. Model-based vs data-driven control of complex networks (poster)

S. Zhao and F. Pasqualetti, "Networks with Diagonal Controllability Gramians: Analysis, Graphical Conditions, and Design Algorithms," *Automatica*, 2019.

F. Pasqualetti, S. Zampieri, F. Bullo, "Controllability Metrics, Limitations and Algorithms for Complex Networks," *IEEE Transactions on Control of Network Systems*, 2014.

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G. Bianchin, F. Pasqualetti, S. Zampieri, "The Role of Diameter in the Controllability of Complex Networks," *IEEE Conf. on Decision and Control*, 2015.

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Acknowledgments

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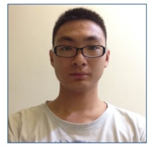

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Controllability, Fragility, and Tradeoffs in Complex Networks

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Analysis and Control of Complex Networks
Philadelphia, PA, July 9 2019