Data-Driven Minimum-Energy Controls for Linear Systems

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***Abstract*—In this letter, we study the problem of computing minimum-energy controls for linear systems from experimental data. The design of open-loop minimum-energy control inputs to steer a linear system between two different states in finite time is a classic problem in control theory, whose solution can be computed in closed form using the system matrices and its controllability Gramian. Yet, the computation of these inputs is known to be ill-conditioned, especially when the system is large, the control horizon long, and the system model uncertain. Due to these limitations, open-loop minimum-energy controls and the associated state trajectories have remained primarily of theoretical value. Surprisingly, in this letter, we show that open-loop minimum-energy controls can be learned exactly from experimental data, with a finite number of control experiments over the same time horizon, without knowledge or estimation of the system model, and with an algorithm that is significantly more reliable than the direct model-based computation. These findings promote a new philosophy of controlling large, uncertain, linear systems where data is abundantly available.**

***Index Terms*—Linear systems, optimal control, statistical learning, identification for control, control of networks.**

I. Introduction

CONSIDER the discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where, respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ denote the system and input matrices, and $x : \mathbb{N} \to \mathbb{R}^n$ and $u : \mathbb{N} \to \mathbb{R}^m$ describe the state and input of the system. For a control horizon $T \in \mathbb{N}$ and a desired state x_f, the minimum-energy control problem asks for the input sequence $u(0), \dots, u(T-1)$ with minimum energy that steers the state from x_0 to x_f in T steps, and it can be formulated as

$$\begin{aligned} \min_u \ & \sum_{t=0}^{T-1} \|u(t)\|_2^2, \\ \text{s.t. } & x(t+1) = Ax(t) + Bu(t), \\ & x(0) = x_0, \ x(T) = x_f. \end{aligned} \tag{2}$$

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As a classic result [1], the minimization problem (2) is feasible if and only if $(x_f - A^T x_0) \in \mathrm{Im}(W_T)$, where

$$W_T = \sum_{t=0}^{T-1} A^t B B^\mathsf{T} (A^\mathsf{T})^t \tag{3}$$

is the T-steps controllability Gramian and $\mathrm{Im}(W_T)$ denotes the image of the matrix W_T. Further, the solution to (2) is

$$u^*(t) = B^\mathsf{T} (A^\mathsf{T})^{T-t-1} W_T^\dagger (x_f - A^T x_0), \tag{4}$$

where $W_T^\dagger$ is the Moore–Penrose pseudoinverse of W_T [2].

The controllability Gramian (3) and the minimum-energy control input (4) identify fundamental control limitations for the system (1), and have been extensively used to solve design [3], sensor and actuator placement [4], and control problems [5] for systems and networks. However, besides their theoretical value, the optimal control input (4) is rarely used in practice or even computed numerically because (i) it relies on the perfect knowledge of the system dynamics, (ii) its performance is not robust to model uncertainties, and (iii) the controllability Gramian is typically ill-conditioned, especially when the system is large [5], [6]. This implies that the control sequence (4) is numerically difficult to compute, and that its implementation leads to errors [7]. To the best of our knowledge, efficient and numerically reliable methods to compute minimum-energy control inputs are still lacking.

Paper Contributions: This letter features two main contributions. First, we show that minimum-energy control inputs for linear systems can be computed from data obtained from control experiments with non-minimum-energy inputs, and without knowledge or estimation of the system matrices. Thus, optimal inputs can be learned from *non-optimal* ones, and we provide three different expressions for doing so. Surprisingly, we also establish that a *finite* number of non-optimal control experiments is always sufficient to compute minimum-energy control inputs towards any reachable state. Second, we show that the data-driven computation of minimum-energy inputs is numerically as reliable as the computation of the inputs based on the exact knowledge of the system matrices, and substantially more reliable than using the closed-form expression based on the Gramian. Further, as minor contributions, we (i) derive bounds on the number of required control experiments as a function of the dimension of the system, number of control inputs, and length of the control horizon, (ii) discuss the effect of noisy data on the data-driven expressions, and (iii) extend our data-driven framework to the case of output measurements.