

Stability Conditions for Cluster Synchronization in Networks of Heterogeneous Kuramoto Oscillators

Tommaso Menara, Giacomo Baggio, Danielle S. Bassett, and Fabio Pasqualetti

Abstract—In this paper we study cluster synchronization in networks of oscillators with heterogeneous Kuramoto dynamics, where multiple groups of oscillators with identical phases coexist in a connected network. Cluster synchronization is at the basis of several biological and technological processes; yet the underlying mechanisms to enable cluster synchronization of Kuramoto oscillators have remained elusive. In this paper we derive quantitative conditions on the network weights, cluster configuration, and oscillators' natural frequency that ensure asymptotic stability of the cluster synchronization manifold; that is, the ability to recover the desired cluster synchronization configuration following a perturbation of the oscillators' states. Qualitatively, our results show that cluster synchronization is stable when the intra-cluster coupling is sufficiently stronger than the inter-cluster coupling, the natural frequencies of the oscillators in distinct clusters are sufficiently different, or, in the case of two clusters, when the intra-cluster dynamics is homogeneous. We illustrate and validate the effectiveness of our theoretical results via numerical studies.

Index Terms—Biological neural network, limit cycle, network theory, nonlinear dynamical systems, stability.

I. INTRODUCTION

SYNCHRONIZATION refers broadly to patterns of coordinated activity that arise spontaneously or by design in several natural and man-made systems [1]–[3]. Examples include coherent firing of neuronal populations in the brain [4], coordinated flashing of fireflies [5], flocking of birds [6], exchange of signals in wireless networks [7], consensus in multi-agent systems [8], and power generation in the smart grid [9]. Synchronization enables complex functions: while some systems require complete (or full) synchronization among all the components in order to function properly, others rely on cluster (or partial) synchronization, where different groups exhibit different, yet synchronized, internal behaviors [10].

While studies of full synchronization are numerous and have generated a rich literature, e.g., see [11]–[13], conditions explaining the onset of cluster synchronization and its properties are less well understood. Such conditions are necessary for the analysis and, more importantly, the control of synchronized activity across biological [14]–[16] and technological [17] systems. For instance, a deeper understanding of the mechanisms

enabling cluster synchronization might not only shed light on the nature of the healthy human brain [18], but also enable and guide targeted interventions for patients with neurological disorders, such as epilepsy [19] and Parkinson's disease [20].

We study cluster synchronization in networks of oscillators with Kuramoto dynamics [21], which, despite their apparent simplicity, are particularly suited to represent complex synchronization phenomena in neural systems [22], as well as in many other natural and technological systems [9]. Although our study and modeling choices are guided by the practical need to understand and control patterns of synchronized functional activity in the human brain, as they naturally arise in healthy and diseased populations [23], [24], in this paper we focus on developing the mathematical foundations of a quantitative approach to the analysis and control of cluster synchronization in a weighted network of Kuramoto oscillators. In particular, we derive conditions on the oscillators' coupling and their natural frequencies that guarantee the stability of an arbitrary cluster configuration.

Related work Cluster synchronization, where multiple synchronized groups of oscillators coexist in a connected network, is an exciting phenomenon that has attracted the attention of the physics, dynamical systems, and controls communities, among others. Existing work on this topic has shown that cluster-synchronized states can be linked to the existence of certain network symmetries [25]–[29] or symmetries in the nodes' dynamics [30]. More recently, in [31], [32], the stability of cluster states corresponding to network symmetries is addressed with the Master Stability Function approach [33]. In contrast to this previous work, [34] combines network symmetries with contraction analysis to study the stability of synchronized states. Further studies relating contraction properties and cluster synchronization are conducted in [35], [36]. Finally, control algorithms for cluster synchronization are developed in [37], [38]. To the best of our knowledge, however, the above studies are not applicable to oscillators with Kuramoto dynamics, which we study in this work.

A few papers have studied cluster synchronization of Kuramoto oscillators. Specifically, in [39], [40] the authors provide invariance conditions for an approximate definition of cluster synchronization and for particular types of networks. Invariance of exact cluster synchronization, which is the notion used in this paper, is also studied in [41], [42]. Stability of exact cluster synchronization is investigated in [43] where, however, only the restrictive case of two clusters for identical Kuramoto oscillators with inertia is considered, and in [44], where only implicit and numerical stability conditions are provided. To the best of our knowledge, our work presents

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