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On the Controllability of Isotropic and Anisotropic Networks

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Abstract— **This paper studies the controllability degree of complex networks as a function of the network weights and the location and number of control nodes. We quantify the controllability degree of a network with the worst-case control energy to drive the network to an arbitrary configuration. We show that isotropic networks are difficult to control, as the control energy grows exponentially with the network cardinality when the number of control nodes remains constant. Conversely, we prove that sufficiently anisotropic networks are easy to control, as the control energy is bounded independently of the network cardinality and number of control nodes.**

I. Introduction

Control of a network refers to the possibility of designing localized interventions to enforce a chosen configuration and to promote a desirable global behavior. Most real-world networks in social, biological and technological domains exhibit complex topological features and dynamics, and it remains an outstanding problem to characterize to what extent these networks can be reprogrammed by few control nodes.

In this work we study the controllability degree of a network as a function of its structural properties. In particular, we quantify the controllability degree of a network with the minimum worst-case energy to reach a desired state [1], and we investigate how the controllability degree depends on the isotropic nature of the network. Inspired by studies on mechanical properties of materials, we define a network to be *isotropic* if it allows a (control) signal to propagate equally in all directions, and to be *anisotropic* otherwise. The network weights determine the anisotropic degree of a network. Our study shows that anisotropic networks are easier to control than isotropic networks, and that the controllability degree may scale differently with the network dimension depending on the degree of anisotropy. As many real-world systems feature anisotropic structures, our results support the thesis that certain complex networks may be efficiently controlled by few carefully selected control nodes [2].

Related work The notion of controllability of dynamical systems (see [1], [3]) has found renewed interest in the context of complex networks, where classic methods are often inapplicable due to the system dimension, and where a graph-inspired understanding of controllability rather than a matrix-theoretical one is preferable. In [4] controllability of complex networks is addressed from a graph-theoretic perpective by employing tools from structured control theory [3]. As discussed in [5], the approach to controllability undertaken in [4] has several limitations, including the fact that the presented results are *generic* [6], and do not account for the network weights. As we show in this work, networks with the same interconnection structure but different weights may exhibit drastically different controllability properties.

The classic binary notion of controllability proposed in [7] and adopted in most works analyzing controllability of complex networks, including [8], [9], does not characterize the difficulty of the control task. In practice, although a network may be controllable by any single node, the actual control input may not be implementable due to actuator constraints and limitations. Instead, we adopt a quantitative measure of network controllability, and we show how the controllability degree scales with the network dimension depending on the network weights and parameters. Surprisingly, we find that for certain networks the controllability degree is independent of the network cardinality and number of control nodes.

A quantitative approach to network controllability has recently been adopted in [10], [11], [12]. We depart from these works by studying the relation between the controllability degree of a network and its isotropic structure. Finally, the observability problem of complex networks is dual to the controllability problem, and equally important [13], [14].

Paper contributions The main contribution of this paper is to prove that certain complex networks can be controlled with finite energy independently of the network cardinality and number of control nodes. We show that the controllability degree scales differently with the network cardinality depending on a notion of network anisotropy, which we characterize as a function of the network weights. Results are first derived for general networks with a technical assumption on the number and location of control nodes, and then specialized to the case of Toeplitz line networks with single control node. A numerical study shows that our results may hold also when our assumptions are violated, suggesting that anisotropic networks may in fact be easier to control.

II. Controllability of Complex Networks

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $A \in \mathbb{R}^{n \times n}$ be the weighted adjacency matrix of $\mathcal{G}$, and let $\mathcal{K} \subseteq \{1, \dots, n\}$ be the set of control nodes. Let $\text{path}(i, j)$ denote a path on $\mathcal{G}$ from node i to node j, and let $|\text{path}(i, j)|$ be the number of edges of $\text{path}(i, j)$. Define the distance between a subset of nodes $S \subseteq \mathcal{V}$ and the control set $\mathcal{K}$ as

$$\text{dist}(\mathcal{S}, \mathcal{K}) = \min\{|\text{path}(i, j)| \, : \, i \in \mathcal{K}, j \in \mathcal{S}\}.$$

Without affecting generality, we order the nodes according to their distance from the set of control nodes. In particular, we

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define $N \in \mathbb{N}$ so that $\mathcal{V} = \cup_{i=1}^{N} \mathcal{V}_i$, with $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$, and $\text{dist}(\mathcal{V}_i, \mathcal{K}) = i - 1$ for all $i \in \{1, \dots, N\}$. According to the partition $\{\mathcal{V}_1, \dots, \mathcal{V}_N\}$, the adjacency matrix reads as

$$A := \begin{bmatrix} D_1 & B_1 & 0 & \cdots & 0 \\ C_1 & D_2 & B_2 & \cdots & 0 \\ 0 & C_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & D_{N-1} & B_{N-1} \\ 0 & \cdots & \cdots & C_{N-1} & D_N \end{bmatrix}, \tag{1}$$

where $D_i \in \mathbb{R}^{|\mathcal{V}_i| \times |\mathcal{V}_i|}$ for $i \in \{1, \dots, N\}$, $B_i \in \mathbb{R}^{|\mathcal{V}_i| \times |\mathcal{V}_{i+1}|}$ and $C_i \in \mathbb{R}^{|\mathcal{V}_{i+1}| \times |\mathcal{V}_i|}$ for $i \in \{1, \dots, N-1\}$, and $|\mathcal{V}_i|$ denotes the cardinality of the set $\mathcal{V}_i$.

We make the following assumption:

(A1) the matrices C_i are of full row rank, that is, $\text{Ker}(C_i^{\mathsf{T}}) = 0$ for all $i \in \{1, \dots, N-1\}$;

Remark 1: ***(Selection of control nodes)*** Assumption (A1) requires the selection of sufficiently many control nodes to be satisfied. Such a selection is straightforward for certain regular networks (see for instance Fig. 1(a)), and it remains the subject of ongoing research for arbitrary topologies. Observe that a necessary condition for (A1) to be satisfied is that $|\mathcal{V}_{i+1}| \leq |\mathcal{V}_i|$ for all $i \in \{1, \dots, N-1\}$. □

Let the T-steps controllability matrix $\mathcal{C}_{\mathcal{K},T}$ and the T-steps controllability Gramian $\mathcal{W}_{\mathcal{K},T}$ be defined, respectively, by

$$\mathcal{C}_{\mathcal{K},T} := \begin{bmatrix} B_{\mathcal{K}} & AB_{\mathcal{K}} & \cdots & A^{T-1}B_{\mathcal{K}} \end{bmatrix}, \text{ and}$$
$$\mathcal{W}_{\mathcal{K},T} := \mathcal{C}_{\mathcal{K},T}\mathcal{C}_{\mathcal{K},T}^{\mathsf{T}},$$

where $B_{\mathcal{K}} \in \mathbb{R}^{n \times m}$ is the network input matrix. Notice that, due to our partitioning scheme in (1),

$$B_{\mathcal{K}} := \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}. \tag{2}$$

Let † denote the Moore-Penrose pseudoinverse operation [15], and define the network parameters

$$\begin{aligned} \alpha &:= \max\{\|D_i\|_2 \;:\; i \in \{1, \dots, N\}\}, \\ \beta &:= \max\{\|B_i\|_2 \;:\; i \in \{1, \dots, N-1\}\}, \text{ and} \\ \gamma &:= \max\{\|C_i^{\dagger}\|_2 \;:\; i \in \{1, \dots, N-1\}\}. \end{aligned} \tag{3}$$

We next characterize the controllability degree of a network as a function of the network parameters α, β and γ. Let $\lambda_{\min}(\mathcal{W}_{\mathcal{K},T})$ be the smallest eigenvalue of the T-steps controllability Gramian, and recall that $\lambda_{\min}^{-1}(\mathcal{W}_{\mathcal{K},T})$ is a tight bound for the minimum control energy to reach any unit-norm state [5] from the zero state.

Theorem 2.1: ***(Controllability degree of block tridiagonal networks)*** Consider a network $\mathcal{G}$ with partition $\{\mathcal{V}_1, \dots, \mathcal{V}_N\}$ and parameters α, β and γ defined in (3).

(i) If $\|A\|_2 < 1$, then for all control sets $\mathcal{K}$ and $T \in \mathbb{N}$

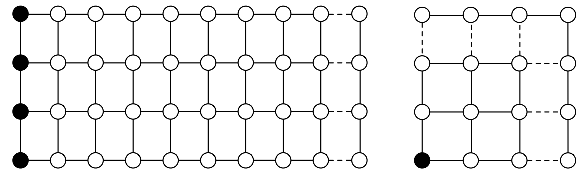
$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \Omega^{\frac{n}{|\mathcal{K}|}},$$

where $\Omega \in \mathbb{R}_{>0}$ is a constant depending only on $\|A\|_2$.

(ii) If $\gamma(1 + \alpha + \beta) < 1$, then for the control set $\mathcal{K} = \{1, \dots, |\mathcal{V}_1|\}$ and for all $T \in \mathbb{N}_{\geq N}$

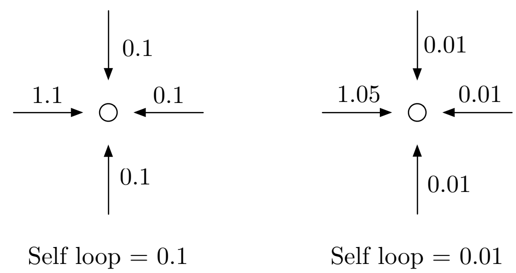
$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \geq \Psi,$$

where $\Psi \in \mathbb{R}_{>0}$ is a constant depending only on α, β, and γ.



(a) Rectangular grid (b) Square grid

Fig. 1. Fig. 1(a) and Fig. 1(b) show the asymmetric rectangular and square grid networks considered in Example 1. Control nodes are marked in black. Network weights are specified in Fig. 2. Fig. 3(a) show the controllability degree of the rectangular grid as the number of column increases. Fig. 3(b) show the controllability degree of the square grid as its dimension increases.



(a) (b)

Fig. 2. Fig. 2(a) and Fig. 2(b) show the edges weight of the networks in Fig. 1(a) and Fig. 1(b), respectively. Weights in the network in Fig. 1(b) are adjusted so that there exists a path from the control node through all other nodes, where the incoming edge of each non-control node in the path has value 1.05. Notice that both networks are strongly anisotropic, as signals do not propagate uniformly throughout the network.

A proof of Theorem 2.1 is postponed to the Appendix. Theorem 2.1 implies that if $\|A\|_2 < 1$, then the controllability degree decreases exponentially with the network cardinality. This behavior is not surprising, and in fact it is also highlighted in [5] for all symmetric networks and for certain asymmetric networks. On the other hand, if the network parameters satisfy the condition $\gamma(1 + \alpha + \beta) < 1$, then the controllability degree admits a positive lower bound, independent of the network cardinality and number of control nodes. Because the controllability degree of symmetric networks decreases exponentially with the network cardinality when the number of control nodes remains constant [5], Theorem 2.1 implies that certain asymmetric networks are easier to control than symmetric networks.

Example 1: ***(Controllability degree of rectangular grids)*** Consider the rectangular grid in Fig. 1(a). Let the network weights be as specified in Fig. 2(a), and let the control nodes be the nodes at the left side of the grid. It can be verified that the network in Fig. 1(a) satisfies assumption (A1) and condition (ii) in Theorem 2.1. As shown in Fig. 3(a), the smallest eigenvalue of the controllability Gramian admits a positive lower bound independent of the network cardinality.

Consider now the square grid in Fig. 1(b). Let the network weights be as specified in Fig. 2(b), and assume the presence of 1 control node as illustrated in Fig. 1(b). It can be verified that assumption (A1) is not satisfied. Yet, as shown in Fig.

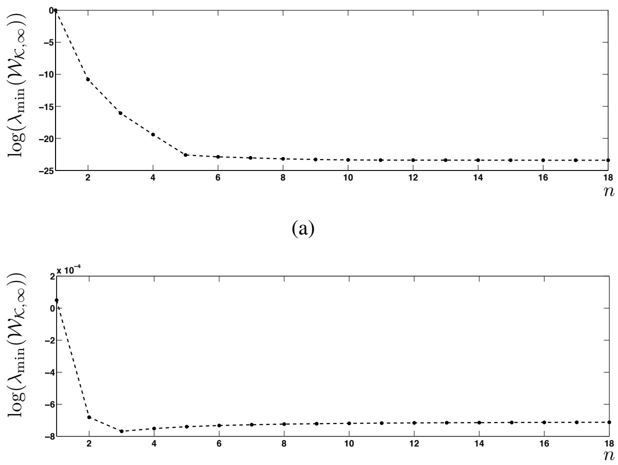


Fig. 3. Fig. 3(a) shows the controllability degree of the rectangular network in Fig. 1(a) with $m = 5$ rows and $n \in \{1, \dots, 25\}$ columns. Fig. 3(b) shows the controllability degree of the square grid in Fig. 1(b) of dimension $n \in \{1, \dots, 10\}$. Network weights are specified in Fig. 2. As predicted by Theorem 2.1 for rectangular grid networks, the controllability degree admits a positive lower bound independent of the network cardinality.

3(b), the weights in Fig. 2(b) numerically ensure that the smallest eigenvalue of the controllability Gramian admits a positive lower bound independent of the network size. □

We next investigate the case of Toeplitz line networks. Let $\mathcal{G}_\ell = (\mathcal{V}_\ell, \mathcal{E}_\ell)$ be a Toeplitz line network, where $\mathcal{V}_\ell = \{1, \dots, n\}$ and $(i, j) \in \mathcal{E}$ whenever $|i - j| \leq 1$. Let A_ℓ be the weighted adjacency matrix of $\mathcal{G}_\ell$, and assume that A_ℓ is a Toeplitz matrix [15] ordered as in (1) with $N = n$, $D_i = a$, $B_i = b$, and $C_i = c$ for all indices i. For this partitioning to be possible, the network contains only one control node, which coincides with node 1 or node n.

Theorem 2.2: ***(Controllability degree of Toeplitz line networks)*** Consider a Toeplitz line network $\mathcal{G}_\ell$ with weights $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{>0}$. Assume $a < 1$.

(i) If one of the following condition is satisfied:

a) $a + b + c < 1$,

b) $ab < c < \frac{b}{a}$ and $a \geq \sqrt{bc}$,

c) $ba^{n_a} < c < \frac{b}{a^{n_a}}$ and $a < \sqrt{bc}$ with $n_a = \frac{2}{\pi} \arccos\left(\frac{-a}{\sqrt{bc}}\right) - 1$,

then for all control sets $\mathcal{K}$ and $T \in \mathbb{N}$

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \Omega^{\frac{n}{|\mathcal{K}|}},$$

where $\Omega \in \mathbb{R}_{>0}$ is a constant depending only on a, b, and c.

(ii) If one of the following condition is satisfied:

a) $\frac{a(b+c)}{4bc} \leq 1$ and $1 < (b - c)^2(1 - \frac{a^2}{4bc})$,

b) $\frac{a(b+c)}{4bc} > 1$ and $1 \leq c + b - a$,

then for the control set $\mathcal{K} = \{1\}$ or $\mathcal{K} = \{n\}$ and for all $T \in \mathbb{N}_{\geq n}$

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \geq \Psi,$$

where $\Psi \in \mathbb{R}_{>0}$ is a constant depending only on a, b, and c.

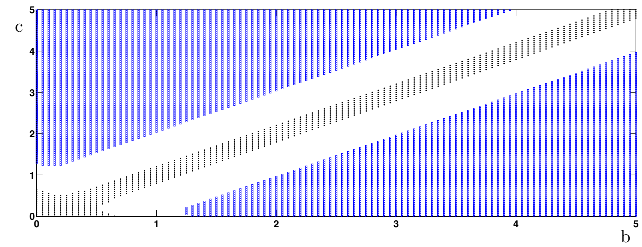


Fig. 4. This figure shows the controllability degree of Toeplitz line networks with $a = 0.3$ as a function of the network weights b and c (see Theorem 2.2). The region identified by blue squares includes networks with bounded controllability degree. The region identified by black dots includes networks whose controllability degree decreases exponentially with the network cardinality. The characterization of the controllability degree of networks outside the two regions remains an outstanding problem.

A proof of Theorem 2.2 is postponed to the Appendix. The conditions in Theorem 2.2 can be visualized in the parameters space. See Fig. 4 for a map of the controllability degree of Toeplitz line networks. It should be observed that the parameter c must be sufficiently larger than the parameter b for the network to feature a bounded controllability degree. Such networks are anisotropic, as (control) signals do not propagate uniformly along all directions. If the degree of anisotropy is not sufficiently large, (control) signals are attenuated along the network, and the controllability degree decreases with the network cardinality. We conclude this section by specializing Theorem 2.2 to the case $a = 0$, which is illustrated in Fig. 5. Fig. 5 suggests that a sharp *controllability transition* from networks with bounded controllability degree to networks with exponentially decaying controllability degree may occur in certain regions of the parameters space. It remains an outstanding problem to formally prove that this controllability transition occurs.

Corollary 2.3: ***(Controllability transition for Toeplitz line networks with*** $a = 0$***)*** Consider a Toeplitz line network $\mathcal{G}_\ell$ with weights $a = 0$, $b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{>0}$.

(i) If $c < 1 - b$ or $c \leq b$, then for all control sets $\mathcal{K}$ and $T \in \mathbb{N}$

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \Omega^{\frac{n}{|\mathcal{K}|}},$$

where $\Omega \in \mathbb{R}_{>0}$ is a constant depending only on b and c.

(ii) If $(b - c)^2 > 1$, then for either the control set $\mathcal{K} = \{1\}$ or $\mathcal{K} = \{n\}$ and for all $T \in \mathbb{N}$

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \geq \Psi,$$

where $\Psi \in \mathbb{R}_{>0}$ is a constant depending only on b and c.

III. Conclusion

In this paper we characterize the controllability degree of a network as a function of the network weights. We show that the controllability degree of networks with identical topologies but different weights may scale differently with

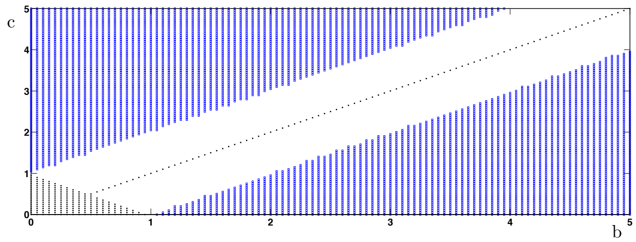


Fig. 5. This figure shows the controllability degree of Toeplitz line networks with $a = 0$ as a function of the network parameters b and c (see Corollary 2.3). The region identified by blue squares includes networks with bounded controllability degree. The region identified by black dots identifies networks whose controllability degree decreases exponentially with the network cardinality. This numerical study suggests that a controllability transition may occur at the points $b = 0$, $c = 1$ and $b = 1$, $c = 0$.

the network cardinality, and this transition may be sharp in the network parameters space. Our results imply that certain networks are controllable with finite energy independently of the network dimension and number of control nodes.

APPENDIX

A. Proof of Theorem 2.1

For the ease of notation, in what follows, we let each submatrix indexed with a nonpositive subscript equal the zero matrix of appropriate dimension. Let $N = T$.

Lemma 3.1: ***(Block-triangular controllability matrix)*** The T-steps controllability matrix can be written as

$$\mathcal{C}_{\mathcal{K},T} := \begin{bmatrix} E_{1,1} & E_{1,2} & \cdots & E_{1,T} \\ 0 & E_{2,2} & \cdots & E_{2,T} \\ \vdots & \ddots & \ddots & E_{T-1,T} \\ 0 & \cdots & 0 & E_{T,T} \end{bmatrix},$$

where $E_{1,1} = I$, and

$$E_{i,j} = C_{i-1}E_{i-1,j-1} + D_i E_{i,j-1} + B_i E_{i+1,j-1}. \quad \text{(A-1)}$$

Proof: Due to the definition of $\mathcal{C}_{\mathcal{K},T}$ we have

$$E(:,j) = AE(:,j-1),$$

where $E(:,j)$ denotes the j-th (block) column of $\mathcal{C}_{\mathcal{K},T}$, and

$$E_{i,j} = A(i,:)E(:,j-1),$$

where $A(i,:)$ the i-th (block) row of A. Since A is block tridiagonal (see equation (1)), the statement follows. ∎

We next show that the T-steps controllability matrix is of full row rank.

Lemma 3.2: ***(Rank of controllability matrix)*** The T-steps controllability matrix is of full row rank, that is,

$$\operatorname{Ker}(\mathcal{C}_{\mathcal{K},T}^{\mathsf{T}}) = 0.$$

Proof: Notice that $E_{1,1} = I$ and $E_{i,i} = \prod_{j=i-1}^{0} C_j$ for $i \in \{2,\dots,T\}$. Because C_j is of full row rank due to assumption (A1), the matrix $E_{i,i}$ is also of full row rank. To conclude the proof, notice that $\mathcal{C}_{\mathcal{K},T}$ is block triangular, where each block is of full row rank. ∎

From Lemma 3.1, the T-steps controllability matrix admits a right inverse, that is, there exists a matrix $Z_{\mathcal{K},T} \in \mathbb{R}^{T|\mathcal{K}|\times n}$ satisfying $\mathcal{C}_{\mathcal{K},T} Z_{\mathcal{K},T} = I$.

Lemma 3.3: ***(Right inverse of $\mathcal{C}_{\mathcal{K},T}$)*** Let

$$\mathcal{Z}_{\mathcal{K},T} := \begin{bmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,T} \\ 0 & G_{2,2} & \cdots & G_{2,T} \\ \vdots & \ddots & \ddots & G_{T-1,T} \\ 0 & \cdots & 0 & G_{T,T} \end{bmatrix},$$

where $G_{1,1} = I$, and

$$G_{i,j} = \left(G_{i-1,j-1} - G_{i,j-1}D_{j-1} - G_{i,j-2}B_{j-2}\right) C_{j-1}^{\dagger}. \quad \text{(A-2)}$$

The matrix $\mathcal{Z}_{\mathcal{K},T}$ satisfies $\mathcal{C}_{\mathcal{K},T}\mathcal{Z}_{\mathcal{K},T} = I$.

Proof: Let $M(:,j)$ and $M(j,:)$ denote the j-th column and the j-th row of the matrix M, respectively.

We proceed by induction. Notice that

$$\mathcal{C}_{\mathcal{K},T}(1,:)\mathcal{Z}_{\mathcal{K},T}(:,1) = I, \quad \mathcal{C}_{\mathcal{K},T}(i,:)\mathcal{Z}_{\mathcal{K},T}(:,1) = 0,$$

for all $i \in \{1,\dots,T\}$. Assume now that

$$\mathcal{C}_{\mathcal{K},T}(k,:)\mathcal{Z}_{\mathcal{K},T}(:,k) = I, \quad \mathcal{C}_{\mathcal{K},T}(i,:)\mathcal{Z}_{\mathcal{K},T}(:,k) = 0, \quad \text{(A-3)}$$

for all $i \in \{1,\dots,T\}$, $i \neq k$, and $k \in \{1,\dots,j-1\}$. We need to show that

(i) $\mathcal{C}_{\mathcal{K},T}(i,:)\mathcal{Z}_{\mathcal{K},T}(:,j) = 0$ for all $i \neq j$, and
(ii) $\mathcal{C}_{\mathcal{K},T}(j,:)\mathcal{Z}_{\mathcal{K},T}(:,j) = I$.

(Case $i > j$): Since $\mathcal{Z}_{\mathcal{K},T}$ is block triangular, it holds $E_{k,j} = 0$ and $G_{k,j} = 0$ for all $k > j$. Then,

$$\mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j) = \sum_{k=1}^{T} E_{i,k}G_{k,j} = \sum_{k=i}^{j} E_{i,k}G_{k,j} = 0.$$

(Case $i = j$): Since $\mathcal{C}_{\mathcal{K},T}$ and $\mathcal{Z}_{\mathcal{K},T}$ are block triangular, we have

$$\mathcal{C}_{\mathcal{K},T}(j,:)\mathcal{Z}_{\mathcal{K},T}(:,j) = \sum_{k=1}^{T} E_{i,k}G_{k,j} = E_{j,j}G_{j,j} = \left(\prod_{k=j}^{1} C_k\right)\left(\prod_{k=1}^{j} C_k^{\dagger}\right) = I,$$

where the last equality follows from the fact that $\operatorname{Ker}(C_i^{\mathsf{T}}) = 0$ for all $i \in \{1,\dots,T\}$, and, consequently, $C_i C_i^{\dagger} = I$ [15].

(Case $i < j$): By using equation (A-2) we obtain

$$\begin{aligned} \mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j) &= \sum_{k=1}^{j} E_{i,k}G_{k,j} \\ &= \sum_{k=i}^{j} E_{i,k}G_{k-1,j-1}C_{j-1}^{\dagger} - \sum_{k=i}^{j} E_{i,k}G_{k,j-1}D_{j-1}C_{j-1}^{\dagger} \\ &- \sum_{k=i}^{j} E_{i,k}G_{k,j-2}B_{j-2}C_{j-1}^{\dagger} = \sum_{k=i}^{j} E_{i,k}G_{i-1,k-1}C_{k-1}^{\dagger} \\ &- \mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j-1)D_{j-1}C_{j-1}^{\dagger} \\ &- \mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j-2)B_{j-2}C_{j-1}^{\dagger}. \end{aligned} \quad \text{(A-4)}$$

Due to equation (A-1) we have

$$\sum_{k=i}^{j} E_{i,k}G_{k-1,j-1}C_{j-1}^{\dagger} = \sum_{k=i}^{j} C_{i-i}E_{i-1,k-1}G_{k-1,j-1}C_{j-1}^{\dagger}$$
$$+ D_iE_{i,k-1}G_{k-1,j-1}C_{j-1}^{\dagger} + B_iE_{i+1,k-1}G_{k-1,j-1}C_{j-1}^{\dagger}$$
$$= C_{i-1}\mathcal{C}_{\mathcal{K},T}(i-1,:)Z_{\mathcal{K},T}(:,j-1)C_{j-1}^{\dagger}$$
$$+ D_i\mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j-1)C_{j-1}^{\dagger}$$
$$+ B_i\mathcal{C}_{\mathcal{K},T}(i+1,:)Z_{\mathcal{K},T}(:,j-1)C_{j-1}^{\dagger}. \tag{A-5}$$

Let $i < j-2$. Due to the hypothesis (A-3) we obtain

$$\mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j) = 0,$$

because each term in equations (A-4) and (A-5) vanishes.

Let $i = j-2$. Due to the hypothesis (A-3) we obtain

$$\mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j) = B_{j-2}C_{j-1}^{\dagger} - B_{j-2}C_{j-1}^{\dagger} = 0.$$

Let $i = j-1$, due to the hypothesis (A-3) we obtain

$$\mathcal{C}_{\mathcal{K},T}(i,:)Z_{\mathcal{K},T}(:,j) = D_{j-1}C_{j-1}^{\dagger} - D_{j-1}C_{j-1}^{\dagger} = 0.$$

This concludes the proof. ■

The matrix $\mathcal{Z}_{\mathcal{K},T}$ is a generalized inverse of $\mathcal{C}_{\mathcal{K},T}$ and, in general, it does not coincide with the Moore-Penrose pseudoinverse [15]. If $\mathcal{C}_{\mathcal{K},T}$ is square, then $\mathcal{Z}_{\mathcal{K},T} = \mathcal{C}_{\mathcal{K},T}^{-1}$.

Lemma 3.4: ***(Lower bound controllability Gramian)*** Let $\mathcal{W}_{\mathcal{K},T}$ be the T-steps controllability Gramian. Then, for all $\bar{T} \geq T$ it holds

$$\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},\bar{T}}) \geq \lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T}) \geq \|Z_{\mathcal{K},T}\|^{-2},$$

where $Z_{\mathcal{K},T}$ is defined in Lemma 3.3.

Proof: Let $x_{\text{f}} \in \mathbb{R}^n$, with $\|x_{\text{f}}\| = 1$ be the network target state, and let $u_{\mathcal{K}}^*(x_{\text{f}})$ be the minimum-energy input to drive the network to the state x_{f} in T steps. Then, $x_{\text{f}} = \mathcal{C}_{\mathcal{K},T}u_{\mathcal{K}}^*(x_{\text{f}})$, and $\|u_{\mathcal{K}}^*(x_{\text{f}})\|_2 \leq \|\mathcal{Z}_{\mathcal{K},T}\|_2$. We conclude that

$$\max_{\|x_{\text{f}}\|=1} \text{E}(u_{\mathcal{K}}^*(x_{\text{f}}), T) = \max_{\|x_{\text{f}}\|=1} \|u_{\mathcal{K}}^*(x_{\text{f}})\|_2^2 = \lambda_{\text{min}}^{-1}(\mathcal{W}_{\mathcal{K},T}) \leq \|\mathcal{Z}_{\mathcal{K},T}\|^2,$$

and the claimed inequality follows. ■

We next derive an upper bound on $\|\mathcal{Z}_{\mathcal{K},T}\|_2$.

Lemma 3.5: ***(Norm of $\mathcal{Z}_{\mathcal{K},T}$)*** Let α, β, and γ be as in (3), and assume that $\gamma(1+\alpha+\beta) < 1$. Then,

$$\|\mathcal{Z}_{\mathcal{K},T}\|_2 \leq c,$$

for some constant $c \in \mathbb{R}$.

Proof: From Lemma 3.3 we have

$$\|\mathcal{Z}_{\mathcal{K},T}(:,j)\|_2 \leq \Bigg\| \Big(z\mathcal{Z}_{\mathcal{K},T}(:,j-1) - \mathcal{Z}_{\mathcal{K},T}(:,j-1)D_{j-1} - \mathcal{Z}_{\mathcal{K},T}(:,j-2)B_{j-2} \Big) C_{j-1}^{\dagger} \Bigg\|_2,$$

where z denotes the shift operator. By using the facts $\|M_1 + M_2\|_2 \leq \|M_1\|_2 + \|M_2\|_2$ and $\|M_1M_2\|_2 \leq \|M_1\|_2\|M_2\|_2$ for every pair of matrices M_1 and M_2 [15], we obtain

$$\begin{bmatrix} \|\mathcal{Z}_{\mathcal{K},T}(:,j+1)\|_2 \\ \|\mathcal{Z}_{\mathcal{K},T}(:,j)\|_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} \gamma(1+\alpha) & \gamma\beta \\ 1 & 0 \end{bmatrix}}_{H} \begin{bmatrix} \|\mathcal{Z}_{\mathcal{K},T}(:,j)\|_2 \\ \|\mathcal{Z}_{\mathcal{K},T}(:,j-1)\|_2 \end{bmatrix}.$$

The characteristic polynomial of H is $P_H(z) = z^2 - \gamma(1+\alpha)z - \gamma\beta$. From Jury's stability test [16], the matrix H is Schur stable if the following three inequalities are satisfied:

$$\begin{cases} 0 & < 1 - \gamma(1+\alpha) - \gamma\beta, \\ 0 & < 1 + \gamma(1+\alpha) - \gamma\beta, \\ 1 & > \gamma\beta. \end{cases}$$

Since $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$, we conclude that H is Schur stable if $\gamma(1+\alpha+\beta) < 1$, in which case the norm of the column of $\mathcal{Z}_{\mathcal{K},T}$ decreases exponentially. To conclude notice that

$$\|\mathcal{Z}_{\mathcal{K},T}\|_2 \leq \|\mathcal{Z}_{\mathcal{K},T}\|_{\text{F}} = \left(\sum_{j=1}^{T} \|\mathcal{Z}_{\mathcal{K},T}(:,j)\|_2^2 \right)^{1/2} \leq c,$$

for some constant $c \in \mathbb{R}_{>0}$. ■

We are now ready to prove Theorem 2.1.

Proof: Let $T_{\text{max}} = \left\lceil \frac{n}{|\mathcal{K}|} \right\rceil - 1$, and notice that the matrix

$$\sum_{\tau=0}^{T_{\text{max}}-1} A^\tau B_{\mathcal{K}} B_{\mathcal{K}}^{\mathsf{T}} A^{\mathsf{T}\tau} = \mathcal{C}_{\mathcal{K},T_{\text{max}}} \mathcal{C}_{\mathcal{K},T_{\text{max}}}^{\mathsf{T}}$$

is singular, where $\mathcal{C}_{\mathcal{K},T_{\text{max}}}$ is the controllability matrix of $(A, B_{\mathcal{K}})$ at T_{max} steps. In fact, $\mathcal{C}_{\mathcal{K},T_{\text{max}}} \in \mathbb{R}^{n\times m}$ with

$$m = T_{\text{max}}|\mathcal{K}| < \left(\frac{n}{|\mathcal{K}|} + 1 \right) |\mathcal{K}| - |\mathcal{K}| = n.$$

An application of the Bauer-Fike theorem [17] for the location of eigenvalues of perturbed matrices yields

$$\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T}) \leq \left(\lambda_{\text{min}} \left(\sum_{\tau=0}^{T-1} A^\tau B_{\mathcal{K}} B_{\mathcal{K}}^{\mathsf{T}} A^{\mathsf{T}\tau} \right) \right)$$
$$\leq \lambda_{\text{min}} \left(\sum_{\tau=0}^{T_{\text{max}}-1} A^\tau B_{\mathcal{K}} B_{\mathcal{K}}^{\mathsf{T}} A^{\mathsf{T}\tau} \right) + \left\| \sum_{\tau=T_{\text{max}}}^{T-1} A^\tau B_{\mathcal{K}} B_{\mathcal{K}}^{\mathsf{T}} A^{\mathsf{T}\tau} \right\|_2$$
$$\leq \sum_{\tau=T_{\text{max}}}^{T-1} \|A\|_2^{2\tau} \|B_{\mathcal{K}}\|_2^2 \leq \frac{\mu^{2(\lceil \frac{n}{|\mathcal{K}|} \rceil - 1)}}{1-\mu^2},$$

where we have used the fact $\|A\|_2 := \mu < 1$. This concludes the proof of statement (i). Statement (ii) follows from Lemma 3.4 and Lemma 3.5. ■

B. Proof of Theorem 2.2

We start with the following instrumental result.

Lemma 3.6: ***(Eigenvalues and condition number of Toeplitz line networks)*** Consider a Toeplitz line network $\mathcal{G}_\ell$ with weights $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{>0}$. The eigenvalues of A_ℓ satisfy

$$\text{spec}(A_\ell) = \left\{ a + 2\sqrt{bc}\cos\left(\frac{k\pi}{n+1}\right) \,:\, k \in \{1,\dots,n\} \right\}.$$

Moreover, there exists an eigenvector matrix V satisfying

$$\text{cond}(V) := \|V\|_2\|V^{-1}\|_2 = \begin{cases} \left(\frac{b}{c}\right)^{\frac{n-1}{2}} & \text{if } b \geq c, \\ \left(\frac{c}{b}\right)^{\frac{n-1}{2}} & \text{if } b < c. \end{cases}$$

Proof: The first statement is a known result; see for instance [18, Lemma 1.77]. To show the second statement, define the diagonal matrix D with diagonal elements δ_i, where

$$\delta_1 = 1, \quad \text{and} \quad \delta_{i+1} = \delta_i \sqrt{\frac{c}{b}}, \quad \text{for } i \in \{1, \dots, n-1\}.$$

Notice that $D^{-1} A_\ell D = \tilde{A}_\ell$, where $\tilde{A}_\ell$ is tridiagonal, Toeplitz and symmetric. Let $\tilde{V}$ be the orthonormal matrix of the eigenvectors of $\tilde{A}_\ell$. Then, $V = D\tilde{V}$ is an eigenvector matrix of A_ℓ. Notice that

$$\text{cond}(V) = \|D\tilde{V}\|_2 \|\tilde{V}^{-1} D^{-1}\|_2 \leq \|D\|_2 \|D^{-1}\|_2,$$

where we have used the fact that $\tilde{V}$ is orthonormal. The statement follows from $\|D\|_2 = \max\{\delta_i : i \in \{1, \dots, n\}\}$ and $\|D^{-1}\|_2 = 1/\min\{\delta_i : i \in \{1, \dots, n\}\}$. ■

We are now ready to prove Theorem 2.2.

Proof: Statement (i) part a) follows from Theorem 2.1. In fact, the condition $a + b + c < 1$ ensures $\|A_\ell\|_\infty = \|A_\ell\|_1 < 1$ and consequently $\|A_\ell\|_2 < 1$.

To show statement (i) part b) and c), recall from [5, Theorem 3.1] that

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \text{cond}^2(V) \frac{\mu^{2(\lceil \frac{n_\mu}{|\mathcal{K}|} \rceil - 1)}}{1 - \mu^2}, \tag{A-6}$$

for all $T \in \mathbb{N}_{>0}$ and for all $\mu \in [\lambda_{\min}(A_\ell), 1)$, where

$$n_\mu = |\{\lambda : \lambda \in \text{spec}(A_\ell), |\lambda| \leq \mu\}|,$$

and V is an eigenvector matrix of the adjacency matrix A_ℓ.

Let $a \geq \sqrt{bc}$ and $\mu = a$. From Lemma 3.6 we have that $n_a \approx n/2$.[1] Lemma 3.6 and equation (A-6) yield

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \begin{cases} \left(\frac{b}{c}\right)^{n-1} \frac{a^{n-2}}{1-a^2} & \text{if } b \geq c, \\ \left(\frac{c}{b}\right)^{n-1} \frac{a^{n-2}}{1-a^2} & \text{if } b < c. \end{cases}$$

Notice that, if $ac < b$, then $\lambda_{\min}(\mathcal{W}_{\mathcal{K},T})$ decreases exponentially with the network cardinality.

Let $0 < a < \sqrt{bc}$ and $\mu = a$. It can be shown that

$$n_a \approx \left(\frac{1}{\pi} \arccos\left(\frac{-a}{\sqrt{bc}}\right) - \frac{1}{2}\right) n. \tag{A-7}$$

In this case, Lemma 3.6 and equation (A-6) imply that

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \begin{cases} \left(\frac{b}{c}\right)^{n-1} \frac{a^{2n_a-2}}{1-a^2} & \text{if } b \geq c, \\ \left(\frac{c}{b}\right)^{n-1} \frac{a^{2n_a-2}}{1-a^2} & \text{if } b < c. \end{cases}$$

Notice that, if $ca^{2n_a/n} < b$, then $\lambda_{\min}(\mathcal{W}_{\mathcal{K},T})$ decreases exponentially with the network cardinality.

We now proceed with statement (ii). Consider the recursion (A-2) defining the inverse of $\mathcal{C}_{\mathcal{K},n}$:

$$cG_{h,k} = G_{h-1,k-1} - aG_{h,k-1} - bG_{h,k-2}. \tag{A-8}$$

We employ Von Neumann analysis [19] to assess the stability of the recursion (A-8). The amplification factor is

$$H(\theta) = \frac{1}{ce^{j\theta} + a + be^{-j\theta}},$$

with $\theta = \pi k/n$, $k \in \{1, \dots, n\}$. The recursion (A-8) is stable if $|H(\theta)| < 1$ for all $\theta \in \mathbb{R}$. Notice that

$$|H^{-1}(\theta)|^2 = (b-c)^2 + a^2 + 2a(b+c)\cos(\theta) + 4bc\cos^2(\theta).$$

By differentiating $|H^{-1}(\theta)|^2$ and computing its fixed points we find that $|H(\theta)| < 1$ for all $\theta \in \mathbb{R}$ if

$$1 \geq \frac{a(b+c)}{4bc}, \text{ and } 1 < (b-c)^2 \left(1 - \frac{a^2}{4bc}\right),$$

or

$$1 < \frac{a(b+c)}{4bc}, \text{ and } 1 <= c + b - a.$$

Notice that, when the recursion (A-8) is stable, $\|Z_{\mathcal{K},T}\|_2$ admits an upper bound independent of the network cardinality. Statement (iii) then follows from Lemma 3.4. ■

[1]The relation $n_a = n/2$ and equation (A-7) hold with equality in the limit for n to infinity. When n is finite appropriate constants should be included. These constants have been omitted to simplify notation, as they do not affect our conclusions.

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