

Routing Apps May Deteriorate Stability in Traffic Networks: Oscillating Congestions and Robust Information Design

Gianluca Bianchin and Fabio Pasqualetti

Abstract—A fundamental question in traffic theory concerns how drivers behave in response to sudden fluctuations of traffic congestion, and to what extent navigation apps can benefit the overall traffic system in these cases. In this paper, we study the stability of the equilibrium points of traffic networks under real-time app-informed routing. We propose a dynamical routing model to describe the real-time route selection mechanism that is at the core of app-informed routing, and we leverage the theory of passivity in nonlinear dynamical systems to provide a theoretical framework to explain emerging dynamical behaviors in real-world networks. We demonstrate for the first time the existence of oscillatory trajectories due to the general adoption of routing apps, which demonstrate how drivers continuously switch between highways in the attempt of minimizing their travel time to destination. Further, we propose a family of control policies to ensure the asymptotic stability of the equilibrium points, which relies on the idea of regulating the rate at which travelers react to traffic congestion. Illustrative numerical simulations combined with empirical data from highway sensors illustrate our findings.

I. INTRODUCTION

Traffic networks are fundamental components of modern societies, making economic activity possible by enabling the transfer of passengers, goods, and services in a timely and reliable fashion. Despite their economical importance, traffic networks are impaired by the outstanding problem of traffic congestion, which causes the waste of over 3 billions of gallons of fuel each year in the United States [1]. Accompanied by increasing levels of congestion is a drop in the system performance, a phenomenon that increasingly conditions the behavior of its users, forcing travelers to shift the time of their morning commute or to adopt alternative routes that are often undesirable and suboptimal. These alternative routes are increasingly made available by routing apps (such as Google Maps, Inrix, Waze, etc.), which provide reliable minimum-time routing suggestions to the travelers based on real-time congestion information. Despite the unprecedented widespread use of app-based routing systems, a characterization of the impact of these devices on the traffic network for general, possibly capacitated, networks has remained elusive until now.

In this work, we study the emerging dynamical behaviors arising when real-time routing decisions are combined with capacitated traffic networks, i.e., where traffic flows have finite propagation times. Our models allow us to take into account the fact that traffic conditions can change while travelers are traversing the network, and that navigation apps will instantaneously respond by updating the route of each

driver at her next available junction. Differently from standard methods to study congestion-responsive routing, the focus of this work is on the dynamical behavior of the traffic network, rather than on the economic properties of its equilibria. Our results demonstrate that the general adoption of navigation apps will maximize the throughput of flow across the network, thus bringing valuable benefits to the traffic infrastructure. Unfortunately, our findings also demonstrate that navigation apps can deteriorate the stability of the traffic system, and can result in the emergence of undesirable traffic phenomena such as temporal oscillations of traffic congestion. Hence, our results demonstrate that the benefits in the adoption of real-time routing systems come at the cost of increased system fragility, and suggest that adequate information design is necessary to overcome these limitations and ensure robustness.

Related Work. This work brings together and extends two streams of independent literature. On the one hand, dynamical traffic network models have widely been studied after the popularization of the Cell Transmission Model [2]. In this line of research, the main emphasis has been on the development of precise numerical models that capture the behavior of the network in several congestion regimes [3], and on characterizing the properties of the equilibria of the network [4], while considering simplified (often time-invariant) routing models.

On the other hand, the routing decisions of the travelers have been studied by adopting simplified traffic models in the game-theoretic setting of a routing game (see e.g. [5]–[7]). In these models, traffic flows propagate *instantaneously* across the network, and drivers make path choices by minimizing their personal travel times in response to day-to-day observations of congestion [8]. Recently, Evolutionary Game-Theory [9] has been applied to the routing game [10], [11], to capture not only the properties of the equilibria of the system, but also the time evolution of its trajectories. Although these works represent a significant step towards understanding the dynamics of traffic routing, the use of simplified traffic models can only model day-to-day reactions to changes in congestion, and lacks to explain emerging dynamical behaviors observed in the data.

An important attempt to characterize the impact of congestion-dependent routing on the dynamical behavior of traffic are the recent works [12], [13], which are however limited to routing models that are local, where travelers make decisions based on the congestion one-road ahead. Finally, to the best of the author’s knowledge, the pioneering work [14] was one of the few attempts to highlight that simplifications in either the traffic model or the routing model are inadequate to accurately predict traffic patterns, and to demonstrate in simulation that in certain regimes static flow models indicate that routing apps can improve network congestion, whereas dynamical models demonstrate the opposite.

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Contribution. The contribution of this work is fourfold. First, we propose a dynamical decision model to capture the reactions of app-informed travelers in response to traffic congestion. Our model is inspired from evolutionary models (or learning models) in biology and game theory, and captures a setting where routing apps use the observations of other travelers to instantaneously adjust their routing suggestions.

Second, we study the properties of the fixed point of a traffic system where our routing decision model is coupled with a dynamical traffic model. We establish a connection between the properties of the equilibrium points and the notion of Wardrop equilibrium [8]. Our results show that, when app-informed travelers can update their routing preferences at every junction of the network based on the instantaneous congestion, the system admits an equilibrium point that satisfies the Wardrop First Principle. This observation extends Wardrop's practical observations, which were so far limited to scenarios where travelers update their congestion information from day to day.

Third, we characterize the Lyapunov stability of the fixed points of the coupled routing-traffic system. Our analysis relies on the theory of passivity for nonlinear dynamical systems [15], and it shows that the equilibrium points are stable but not always asymptotically stable. Moreover, for a network composed of two parallel highways we show the existence of limit cycles, thus demonstrating that traffic congestion can oscillate over time, a phenomenon that was recently observed in [16].

Fourth, we propose a control technique to ensure the asymptotic stability of the fixed points. Our method relies on regulating the rates at which travelers react to congestion information, a behavior that can be achieved by appropriately designing the frequency at which navigation apps update the routing suggestions provided to the travelers. Our results suggest that, in order to achieve asymptotic stability of the equilibrium points, travelers that are close to the network origin must react faster to traffic congestion as opposed to travelers that are located in the proximity of the destination.

Organization. This paper is organized as follows. Section II illustrates our traffic network model, our routing decision model, and reviews the Wardrop First Principle. Section III characterizes the properties of the equilibrium points, and contains a set of necessary and sufficient conditions for their existence. Section IV contains the stability analysis of the equilibrium points, and illustrates through an example the existence of oscillatory trajectories. Section V proposes a control technique to ensure the asymptotic stability of the fixed points, and Section VI illustrates our findings through a set of simulations. Finally, Section VII summarizes our conclusions, while Appendix A contains a primer on concepts from nonlinear system theory that are relevant to this work.

Notation. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{L})$, consists of a set of vertices \mathcal{V} and a set of directed links (or edges) $\mathcal{L} \subseteq \mathcal{V} \times \mathcal{V}$. We use the notation $\ell = (v, w)$ to denote a directed link from node $v \in \mathcal{V}$ to node $w \in \mathcal{V}$ and, for each node v , we let $v^{\text{out}} = \{(z, w) \in \mathcal{L} : z = v\}$ be the set of its outgoing links and $v^{\text{in}} = \{(w, z) \in \mathcal{L} : z = v\}$ be the set of its incoming links. A path in \mathcal{G} is a subgraph $p = (\{v_1, \dots, v_k\}, \{\ell_1, \dots, \ell_k\})$, such that $v_i \neq v_j$ for all $i \neq j$, and $\ell_i = (v_i, v_{i+1})$ for each $i \in \{1, \dots, k-1\}$. A path p is simple (or edge-disjoint) if

no link is repeated in p . We will say that a path starts at v_1 and ends at v_k , and use the compact notation $p = p_{v_1 \rightarrow v_k}$. A cycle is a path where the first and last vertex are identical, i.e., $v_1 = v_k$. Finally, \mathcal{G} is acyclic if it contains no cycles.

II. TRAFFIC NETWORK AND APP ROUTING MODELS

This section is organized into three main parts. First, we discuss a traffic model that captures the physical characteristics of roads and traffic junctions. Second, we introduce a decision model to capture the routing behavior of app-informed travelers in response to traffic congestion. Third, we review the framework that describes the Wardrop First Principle.

A. Traffic Network Model

We model a traffic network as a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{L})$, where $\mathcal{L} = \{1, \dots, n\} \subseteq \mathcal{V} \times \mathcal{V}$ models the set of traffic roads (or links), and $\mathcal{V} = \{v_1, \dots, v_\nu\}$ models the set of traffic junctions (or nodes). Every traffic junction is composed of a set of ramps, each interconnecting a pair of freeways. We denote the set of traffic ramps (or adjacent links) by $\mathcal{A} \subseteq \mathcal{L} \times \mathcal{L}$ and let \mathcal{A}_ℓ be the set of ramps available upon exiting ℓ , that is,

$$\begin{aligned} \mathcal{A} &:= \{(\ell, m) : \exists v \in \mathcal{V} \text{ s.t. } \ell \in v^{\text{in}} \text{ and } m \in v^{\text{out}}\}, \\ \mathcal{A}_\ell &:= \{m \in \mathcal{L} : \exists (\ell, m) \in \mathcal{A}\}. \end{aligned} \quad (1)$$

We describe the macroscopic behavior of each link $\ell \in \mathcal{L}$ by means of a dynamical equation that captures the conservation of flows between upstream and downstream:

$$\dot{x}_\ell = f_\ell^{\text{in}}(x) - f_\ell^{\text{out}}(x_\ell),$$

where $x_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathcal{X}$, $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}$, is the traffic density in the link, $f_\ell^{\text{in}} : \mathcal{X} \rightarrow \mathcal{F}$, $\mathcal{F} \subseteq \mathbb{R}_{\geq 0}$, is the inflow of traffic at the link upstream, and $f_\ell^{\text{out}} : \mathcal{X} \rightarrow \mathcal{F}$ is the outflow of traffic at the link downstream. We make the following technical assumption.

(A1) For all $\ell \in \mathcal{L}$, $f_\ell^{\text{out}}(x_\ell) = 0$ only if $x_\ell = 0$. Moreover, f_ℓ^{out} is differentiable, non-decreasing, and upper bounded by the flow capacity of the link $C_\ell \in \mathbb{R}_{\geq 0}$:

$$\frac{d}{dx_\ell} f_\ell^{\text{out}}(x_\ell) \geq 0 \text{ and } \sup_{x_\ell} f_\ell^{\text{out}}(x_\ell) = C_\ell.$$

We discuss in the following remark possible choices of outflow functions commonly adopted in practice.

Example 1: (Common Link Outflow Functions) A common choice for the link outflow function is the linear saturation function, originally adopted by the Cell Transmission Model [2], described by

$$f_\ell^{\text{out}}(x_\ell) = \min\{v_\ell x_\ell, C_\ell\},$$

where $v_\ell \in \mathbb{R}_{> 0}$ models the *free-flow speed* of the link. Linear outflow functions have also been considered in the literature thanks to their simplicity [17]:

$$f_\ell^{\text{out}}(x_\ell) = v_\ell x_\ell,$$

where, in this case, $C_\ell = +\infty$. Alternatively, exponential saturation functions have widely been adopted in the recent literature (see e.g. [4]):

$$f_\ell^{\text{out}}(x_\ell) = C_\ell(1 - \exp(-a_\ell x_\ell)),$$

where $a_\ell \in (0, \infty)$. \square

We associate a routing ratio $r_{\ell m} \in [0, 1]$ to every pair of adjacent links $(\ell, m) \in \mathcal{A}$ to describe the fraction of traffic flow entering link m upon exiting ℓ , with $\sum_m r_{\ell m} = 1$. We combine the routing ratios into a matrix $R = [r_{\ell m}] \in \mathbb{R}^{n \times n}$, where we let $r_{\ell m} = 0$ if ℓ and m are not adjacent $(\ell, m) \notin \mathcal{A}$, and we denote by $\mathcal{R}_{\mathcal{G}}$ the set of feasible routing ratios for the network defined by \mathcal{G} . That is,

$$\mathcal{R}_{\mathcal{G}} := \{r_{\ell m} : r_{\ell m} = 0 \text{ if } (\ell, m) \notin \mathcal{A}, \text{ and } \sum_{m \in \mathcal{L}} r_{\ell m} = 1\}.$$

At every ramp, traffic flows are transferred from the incoming link to the outgoing link as described by the routing ratios:

$$f_m^{\text{in}}(x) = \sum_{\ell \in \mathcal{L}} r_{\ell m} f_\ell^{\text{out}}(x_\ell).$$

We focus on single-commodity networks, where an inflow of vehicles $\bar{\lambda} : \mathbb{R}_{\geq 0} \rightarrow \mathcal{F}$ enters the network at a (unique) source link $s \in \mathcal{L}$, and traffic flows exit the network at a (unique) destination link $d \in \mathcal{L}$. In the remainder, we adopt the convention $s = 1$ and $d = n$. We describe the overall network dynamics by combining the dynamical models of all links in a vector equation of the form

$$\dot{x} = (R^T - I)f(x) + \lambda, \quad (2)$$

where $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix, $x = [x_1, \dots, x_n]^T$ is the vector of traffic densities in the links, $f = [f_1^{\text{out}}, \dots, f_n^{\text{out}}]^T$ is the vector of link outflows, and $\lambda = [\bar{\lambda}, \dots, 0]^T$ denotes the inflow vector. Finally, we illustrate our model of traffic network in Example 2, and we discuss the relationship between our model and the well-established Cell Transmission Model in Remark 1.

Example 2: (Dynamical Traffic Model) Consider the seven-link network illustrated in Fig. 1. The traffic network model (2) is composed of the following seven dynamical equations:

$$\begin{aligned} \dot{x}_1 &= -f_1^{\text{out}}(x_1) + \bar{\lambda}, \\ \dot{x}_2 &= -f_2^{\text{out}}(x_2) + r_{12}f_1^{\text{out}}(x_1), \\ \dot{x}_3 &= -f_3^{\text{out}}(x_3) + r_{13}f_1^{\text{out}}(x_1), \\ \dot{x}_4 &= -f_4^{\text{out}}(x_4) + r_{24}f_2^{\text{out}}(x_2), \\ \dot{x}_5 &= -f_5^{\text{out}}(x_5) + r_{25}f_2^{\text{out}}(x_2), \\ \dot{x}_6 &= -f_6^{\text{out}}(x_6) + f_3^{\text{out}}(x_3) + f_4^{\text{out}}(x_4), \\ \dot{x}_7 &= -f_7^{\text{out}}(x_7) + f_5^{\text{out}}(x_5) + f_6^{\text{out}}(x_6), \end{aligned}$$

where

$$\mathcal{R}_{\mathcal{G}} = \{r_{12}, r_{13}, r_{24}, r_{25} : r_{12} + r_{13} = 1, r_{24} + r_{25} = 1\}.$$

\square

Remark 1: (Capturing Backwards Propagation) Our model can be interpreted as a simplified version of the Cell Transmission Model [2]. In fact, while in the Cell Transmission Model highways are characterized by two fundamental functions (a link *demand* function and a link *supply* function), our model only captures capacities in the flows through the link outflow functions f_ℓ^{out} . As a result, in our model density accumulation can happen on the links and congestion does not propagate through the junctions (and thus corresponds to a vertical queue

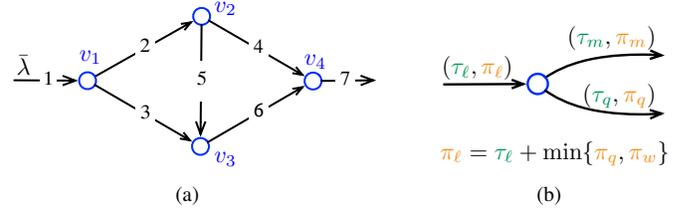


Fig. 1. (a) Seven-highway network discussed in examples 2 and 3. (b) We associate two variables to each link: the link travel cost τ_ℓ (the travel time to traverse that link) and the link perceived cost π_ℓ (the total travel cost of reaching the network destination from that link).

model). While more general traffic models could be considered in future works, we note that density capacities can be captured in our model by considering unbounded link delay functions, as we do in Section III-C. \square

B. Congestion-Responsive Routing Model

In what follows, we present a dynamical decision model to capture the behavior of app-informed travelers in response to congestion. To this aim, we associate a state-dependent travel cost to each link of the network

$$\tau_\ell : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{T} \subseteq \mathbb{R}_{\geq 0},$$

which describes the instantaneous travel cost (or travel delay) of traversing link ℓ . We denote by $\tau(x) = [\tau_1, \dots, \tau_n]^T$ the joint vector of travel costs, and we make the following technical assumption.

(A2) For all $\ell \in \mathcal{L}$, the travel cost $\tau_\ell(x_\ell)$ is differentiable and non-decreasing.

To capture the fact that travelers wish to minimize the overall (total) travel time between their current location and their destination, we associate to each link ℓ a perceived cost:

$$\pi_\ell : \mathcal{X}^n \rightarrow \mathcal{T},$$

which describes the cost of link ℓ that is *perceived* by the travelers. The perceived cost is a quantity that, in general, includes the combined cost of traversing multiple links (e.g. a path in the graph). In this work, we model the perceived costs as the instantaneous minimum travel times to destination (see Fig. 1(b))

$$\pi_\ell(x) = \tau_\ell(x_\ell) + \min_{m \in \mathcal{A}_\ell} \pi_m(x). \quad (3)$$

We note that the above equation is a recursive definition, and: (i) given the current traffic state, the set of perceived costs can be computed backwards from the network destination to every link in the graph, and (ii) the above equation states that a traveler located at any point in the traffic network perceives a cost that is equal to the instantaneous minimum travel time to destination. We discuss and generalize the choice of perceived costs in Remark 2.

Remark 2: (Choices of Perceived Costs) A choice that generalizes (3) is the following convex combination:

$$\pi_\ell(x) = \alpha_\ell \tau_\ell(x_\ell) + (1 - \alpha_\ell) \min_{m \in \mathcal{A}_\ell} \pi_m(x),$$

where $\alpha_\ell \in [0, 1]$ is a parameter that describes the level of confidence in the observed global congestion information. For instance, the special case $\alpha_\ell = 1$ correspond to a situation where the drivers rely only local congestion information, while $\alpha_\ell = 0$ models a scenario where drivers rely on global congestion information, which is the focus of this work. An intermediate value of α_ℓ can be interpreted as the level of confidence in the knowledge of the travel delay of links that are distant in the network. Although all the results presented in this paper hold for the generalized perceived cost model, in the remainder of this paper we focus on the model (3) for the clarity of illustration. \square

To model the reactions of app-informed travelers to changes in the traffic state, we assume that at every node of the network drivers will instantaneously update their routing by increasingly avoiding the links with higher perceived cost (in the current congestion regime). To this aim, we model the aggregate routing ratios as time-varying quantities $r_{\ell m} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ that obey a selection mechanism inspired by the replicator dynamics [9]:

$$\delta_{\ell m}^{-1} \dot{r}_{\ell m} = r_{\ell m} \underbrace{\left(\sum_q r_{\ell q} \pi_q - \pi_m \right)}_{a_{\ell m}(x)}, \quad (4)$$

where $a_{\ell m} : \mathcal{X}^n \rightarrow \mathbb{R}$ is a function that describes the appeal of entering link m upon exiting ℓ , and $\delta_{\ell m} \in \mathbb{R}_{>0}$ is the reaction rate, namely a scalar variable that captures the rate at which travelers react to changes in the traffic state.

The dynamical equation (4) describes a real-time reaction mechanism, where routing apps continuously revise their routing recommendations by increasingly suggesting links that have a more desirable travel time to destination, as detailed next. A positive appeal ($a_{\ell m} > 0$) implies that the perceived travel cost of link m is preferable over the travel cost of alternative links upon exiting ℓ (i.e. $\pi_m < \sum_q r_{\ell q} \pi_q$). Hence, equation (4) states that the fraction of travelers choosing m will increase over time ($\dot{r}_{\ell m} > 0$). As a result, the appeal $a_{\ell m}$ can be interpreted as the aggregate interest in selecting to traverse link m upon exiting ℓ .

In compact form, the set of dynamical equations (4) describing the routing parameters reads as follows:

$$\dot{r} = \varrho(r, \pi), \quad (5)$$

where $r = [\dots, r_{\ell m}, \dots]^T$, $(\ell, m) \in \mathcal{A}$, denotes the joint vector of routing ratios. In the following result, we show that the congestion-responsive routing model (5) evolves within the feasible set of routing ratios $\mathcal{R}_{\mathcal{G}}$ at all times.

Lemma 2.1: (Conservation of Flows) Let \mathcal{G} be a traffic network and let $\delta_{\ell m} = \delta_\ell \in \mathbb{R}_{>0}$ for all $(\ell, m) \in \mathcal{A}$. If $r(0) \in \mathcal{R}_{\mathcal{G}}$, then the vector of routing ratios is feasible at all times, that is,

$$r \in \mathcal{R}_{\mathcal{G}} \text{ for all } t \in \mathbb{R}_{>0}.$$

Proof: The proof of this claim is organized into two parts. First, we show that $r_{\ell m} \in [0, 1]$. To show that the routing ratios are non-negative, $r_{\ell m} \geq 0$, we note that

$$r_{\ell m} = 0 \Rightarrow \dot{r}_{\ell m} = r_{\ell m} a_{\ell m}(x) = 0.$$

To show that the routing ratios are upper bounded, $r_{\ell m} \leq 1$, assume the ratio achieves the boundary, i.e. $r_{\ell m} = 1$. Then, since $r \in \mathcal{R}_{\mathcal{G}}$ (i.e., $\sum_q r_{\ell q} = 1$), we have

$$r_{\ell q} = 0 \text{ for all } q \neq m,$$

which implies

$$a_{\ell m}(x) = \sum_q r_{\ell q} \pi_q - \pi_m = r_{\ell m} \pi_m - \pi_m = 0.$$

Hence, the above observations prove the following implication

$$r_{\ell m} = 1 \Rightarrow \dot{r}_{\ell m} = r_{\ell m} a_{\ell m}(x) = 0,$$

which shows that the routing ratios are bounded in $[0, 1]$.

Second, we prove that $\sum_m r_{\ell m} = 1$. To this aim, we equivalently show that $\sum_m \dot{r}_{\ell m} = 0$. By substituting the expression (4) in the summation term we obtain

$$\begin{aligned} \sum_m \dot{r}_{\ell m} &= \sum_m r_{\ell m} \left(\sum_q r_{\ell q} \pi_q - \pi_m \right) \\ &= \sum_m r_{\ell m} \underbrace{\sum_q r_{\ell q} \pi_q}_{=1} - \sum_m r_{\ell m} \pi_m \\ &= \sum_q r_{\ell q} \pi_q - \sum_m r_{\ell m} \pi_m = 0, \end{aligned}$$

which shows the claim and concludes the proof. \blacksquare

We conclude this discussion by illustrating in Example 3 our routing model, by discussing in Remark 3 the use of the replicator equation to model routing apps, and by clarifying in Remark 4 the novelty of our framework with respect to the classical routing game.

Example 3: (Dynamical Routing Model) Consider the seven-link network illustrated in Fig. 1 and discussed in Example 2. By assuming that the drivers perceive the global cost to destination (3), the perceived costs read as

$$\begin{aligned} \pi_1 &= \tau_1 + \bar{\pi}_{v_1 \rightarrow d}, & \bar{\pi}_{v_1 \rightarrow d} &= \min\{\pi_2, \pi_3\}, \\ \pi_2 &= \tau_2 + \bar{\pi}_{v_2 \rightarrow d}, & \bar{\pi}_{v_2 \rightarrow d} &= \min\{\pi_4, \pi_5\}, \\ \pi_3 &= \tau_3 + \bar{\pi}_{v_3 \rightarrow d}, & \bar{\pi}_{v_3 \rightarrow d} &= \pi_6, \\ \pi_4 &= \tau_4 + \bar{\pi}_{v_3 \rightarrow d}, & & \\ \pi_5 &= \tau_5 + \bar{\pi}_{v_4 \rightarrow d}, & \bar{\pi}_{v_4 \rightarrow d} &= \pi_7, \\ \pi_6 &= \tau_6 + \bar{\pi}_{v_4 \rightarrow d}, & & \\ \pi_7 &= \tau_7. & & \end{aligned}$$

We note that the perceived costs (3) are defined in a recursive way, where for all $i \in \{1, \dots, 7\}$, π_i can be computed given π_{i+1}, \dots, π_7 . Moreover, the aggregate behavior of the population at the nodes is described as in (4) by

$$\begin{aligned} \dot{r}_{12} &= r_{12}((r_{12} - 1)\pi_2 + r_{13}\pi_3), \\ \dot{r}_{13} &= r_{13}((r_{13} - 1)\pi_3 + r_{12}\pi_2), \\ \dot{r}_{24} &= r_{24}((r_{24} - 1)\pi_4 + r_{25}\pi_5), \\ \dot{r}_{25} &= r_{25}((r_{25} - 1)\pi_5 + r_{24}\pi_4). \end{aligned}$$

Finally, we note that Lemma 2.1 ensures $r_{12} + r_{13} = 1$ and $r_{24} + r_{25} = 1$ at all times. \square

Remark 3: (Modeling Aggregate Learning Through Replicator Equation) The replicator equation was originally developed to study selection in biological evolution. However, it was found recently in [18] (see also references therein) that the evolutionary replicator dynamics can also arise from certain models of human learning. Moreover, in a more recent work [19] it was shown that if models of reinforcement learning or other machine learning techniques were aggregated over a large population, the resulting behavior would possess the same qualitative properties as the replicator dynamics. \square

Remark 4: (Relationship to Routing Game) A trend of literature (e.g. see [10], [11]) recently combined the classical routing game with evolutionary models in order to capture dynamics in the path-selection mechanism of new drivers entering the network. Although these works represent a significant step towards understanding the dynamics of traffic routing, they still critically rely on a static flow model, where traffic flows instantaneously propagate across the network. Unfortunately, this assumption lacks to capture the fact that traffic conditions can change while travelers are traversing the network, and that navigation apps will instantaneously respond by updating the route of each driver at her next available junction. To overcome these limitations, our framework (i) leverages a dynamical traffic model that captures finite flow propagation times, and (ii) includes a junction-based routing model where travelers can update their routing behavior at every node of the network in relationship to the current congestion information. \square

C. The Wardrop First Principle

The goal of this section is to establish a connection between the classical game-theoretic setting and our framework. The routing game [8] consists of a static (time-invariant) traffic model combined with a path-selection model. In this decision model, a new traveler entering the network selects a certain origin-destination path based on the instantaneous traffic congestion and, because the traffic model is static, drivers do not update their path while they are traversing the network. Once this path-selection mechanism terminates, the network is at an equilibrium point known as the Wardrop Equilibrium, a condition where all the used paths have identical travel time.

Next, we recall the notion Wardrop Equilibrium. To comply with the static nature of the routing game, we will assume that the dynamical system (7) is at an equilibrium point. Let x^* be an equilibrium of (2), and let

$$f_\ell^* := f_\ell^{\text{out}}(x_\ell^*), \quad \ell \in \mathcal{L},$$

be the set of equilibrium flows on the links. In vector form, $f^* := [f_1^* \dots f_n^*]^\top$. Moreover, let $\mathcal{P} = \{p_1, \dots, p_\zeta\}$, $\zeta \in \mathbb{N}$, be the set of simple paths between origin and destination, and let $f_p^* := [f_{p_1}^*, \dots, f_{p_\zeta}^*]^\top$ be the set of flows on the paths. The flows on the origin-destination paths are related to the flows on the links by means of the following relationship:

$$f_\ell^* = \sum_{p \in \mathcal{P}: \ell \in p} f_p^*,$$

which establishes that the flow on each link is the superposition of all the flows in the paths passing through that link. By

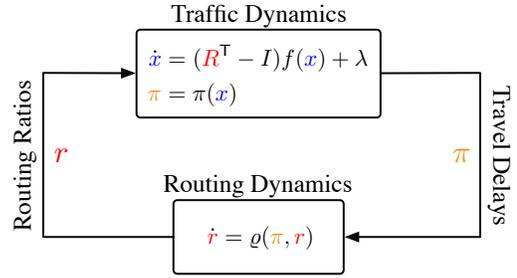


Fig. 2. Feedback interconnection between traffic and routing dynamics.

inverting the above set of equations, the vector of path flows can be computed from the vector of link flows as follows

$$f_p^* = E^\dagger f^*, \quad (6)$$

where $E \in \mathbb{R}^{n \times \nu}$ is the edge-path incidence matrix:

$$E_{\ell p} = \begin{cases} 1, & \text{if } \ell \in p, \\ 0, & \text{otherwise,} \end{cases}$$

where E^\dagger denotes the pseudoinverse of E . Lemma 2.2 shows that the path flows are unique for any choice of link flows.

Lemma 2.2: (Uniqueness of the Path Flow Vectors) Let \mathcal{G} be acyclic. Then, for every vector of link flows $f^* \in \mathcal{F}^n$ there exists a unique vector of path flows $f_p^* \in \mathbb{R}^\zeta$ that solves (6). The proof of this lemma is postponed to the appendix.

We extend the definition of travel costs to the origin-destination paths by letting the travel cost of a path be the sum of the cost of all the links in that path, namely,

$$\tau_p^* := E^\top \tau(x^*).$$

The Wardrop First Principle states that all paths with nonzero flow have identical travel cost, and is formalized next.

Definition 1: (Wardrop First Principle) Let x^* be an equilibrium of (2). The vector x^* is a *Wardrop Equilibrium* if the following condition is satisfied for all origin-destination paths $p \in \mathcal{P}$:

$$f_p^*(\tau_p^* - \tau_{\bar{p}}^*) \leq 0, \quad \text{for all } \bar{p} \in \mathcal{P}.$$

III. EXISTENCE AND PROPERTIES OF THE EQUILIBRIA

In this section, we characterize the properties of the fixed points of dynamical traffic networks with app-informed routing. Formally, we are interested in characterizing the fixed points of the feedback interconnection between the traffic dynamics (2) and the routing dynamics (5), which reads as:

$$\begin{aligned} \dot{x} &= (R^\top - I)f(x) + \lambda, & \pi &= \pi(x), \\ \dot{r} &= \varrho(r, \pi). \end{aligned} \quad (7)$$

Fig. 2 graphically illustrates the interactions between the two systems and depicts the quantities that establish the coupling.

A. Restricted Set of Equilibria

Let (x^*, r^*) be a fixed point of (7). It follows from the expressions of the routing model (4) that, for all pairs of adjacent links $(\ell, m) \in \mathcal{A}$, one of the following conditions is satisfied at equilibrium:

$$a_{\ell m}(x^*) = 0, \text{ or } r_{\ell m}^* = 0.$$

We next show that a subset of these points is unstable.

Lemma 3.1: (Unstable Equilibria) Let (x^*, r^*) be a fixed point of (7) and assume there exists $(\ell, m) \in \mathcal{A}$ such that

$$r_{\ell m}^* = 0, \text{ and } a_{\ell m}(x^*) > 0.$$

Then, (x^*, r^*) is unstable.

Proof: To prove this lemma, we adopt a perturbation reasoning and show that there exists an infinitesimally-small perturbation from the equilibrium such that $\dot{r}_{\ell m} > 0$. The proof is organized into two main parts.

First, we show that at equilibrium all links alternative to m have identical perceived cost. To this aim, we note that $r_{\ell m}^* = 0$ combined with $r^* \in \mathcal{R}_{\mathcal{G}}$ (i.e. $\sum_q r_{\ell q}^* = 1$) implies that there exists (at least) one alternative link w such that $r_{\ell w}^* > 0$. In general, let $\mathcal{W} = \{w_1, \dots, w_{\xi}\}$, $\xi \in \mathbb{N}$, denote the set of all such links. Since $r_{\ell w_i}^* > 0$ and x^* is an equilibrium, we necessarily have $a_{\ell w_i}(x^*) = 0$ or, equivalently,

$$0 = a_{\ell w_i}(x^*) = \sum_q r_{\ell q}^* \pi_q^* - \pi_{w_i}^*,$$

for all $i \in \{1, \dots, \xi\}$. The above system of equations admits the explicit solution $\pi_{w_i}^* = \sum_q r_{\ell q}^* \pi_q^*$ for all $i \in \{1, \dots, \xi\}$, which implies

$$\pi_{w_i}^* = \pi_{w_j}^*, \text{ for all } i, j \in \{1, \dots, \xi\}, \quad (8)$$

and proves the first claim.

Second, we show that at equilibrium all links alternative to m (i.e. links $w \in \mathcal{W}$), have strictly suboptimal travel cost: $\pi_w^* > \pi_m^*$. To this aim, we use the assumption $a_{\ell m}(x^*) > 0$ to obtain

$$\begin{aligned} 0 < a_{\ell m}(x^*) &= \sum_q r_{\ell q}^* \pi_q^* - \pi_m^* \\ &= \pi_w^* \sum_q r_{\ell q}^* - \pi_m^* \\ &= \pi_w^* - \pi_m^*, \end{aligned} \quad (9)$$

where we substituted (4) to obtain the first identity, and (8) to obtain the second identity, which proves the second claim.

Finally, let $\epsilon \in \mathbb{R}_{>0}$ be a scalar perturbation. By perturbing (4) from the equilibrium point, $r_{\ell m}^* \mapsto r_{\ell m}^* + \epsilon$, we have

$$\begin{aligned} \dot{r}_{\ell m} &= \epsilon \left(\sum_{q \neq m, w} r_{\ell q}^* \pi_q^* + (r_{\ell m}^* + \epsilon) \pi_m^* + (r_{\ell w}^* - \epsilon) \pi_w^* - \pi_m^* \right) \\ &= \epsilon \left(\sum_q r_{\ell q}^* \pi_q^* + \epsilon \pi_m^* - \epsilon \pi_w^* - \pi_m^* \right) \\ &= \epsilon \left(\underbrace{\sum_q r_{\ell q}^*}_{=1} \pi_q^* + \epsilon \pi_m^* - \epsilon \pi_w^* - \pi_m^* \right) \\ &= \epsilon (\pi_w^* + \epsilon \pi_m^* - \epsilon \pi_w^* - \pi_m^*) \\ &= \epsilon (\pi_w^* - \pi_m^*) (1 - \epsilon) > 0, \end{aligned}$$

where we used (8) to obtain the third identity, and the final inequality follows from (9) and from $\epsilon > 0$. The conclusion follows by observing that infinitely-small perturbations $\epsilon \rightarrow 0$ result in systems that depart from the equilibria $\dot{r}_{\ell m} > 0$. ■

Lemma 2.1 shows that equilibrium points where at least one of the links has a positive appeal function are unstable. Such scenarios can be interpreted in practice as a situation where there exists a link in the network with a preferable travel time to destination (i.e. $a_{\ell m} > 0$), but no driver is currently traversing that road (i.e. $r_{\ell m} = 0$). Hence, the navigation app lacks of sufficient observations from other travelers to begin routing vehicles towards that road, thus making the routing algorithm ignore the availability of such option.

In order to disregard the unstable equilibria from the discussion, in the remainder we focus on the equilibria (x^*, r^*) such that, for all $(\ell, m) \in \mathcal{A}$, satisfy:

$$a_{\ell m}(x^*) = 0, \text{ or } r_{\ell m} = 0 \text{ and } a_{\ell m}(x^*) < 0. \quad (10)$$

Remark 5: (Relationship to Game Dynamics) The set of equilibria defined in (10) is often interpreted in the game-theoretic literature as the set of Nash Equilibria of the game dynamics (4) (see e.g. [20]). It is worth noting that Lemma 2.1 extends the available results in this line of literature (e.g. see the Folk Theorem of evolutionary game theory [9] and the specific conclusions drawn for the routing game by Fischer and Vöcking [10]), by showing that the set of rest points that are not Nash equilibria are unstable for replicator equations where the payoffs do not depend directly from the strategy. □

B. Existence of Equilibria

Next, we characterize the existence of fixed points of the interconnected system (7). Our result relies on the following technical assumption.

(A3) The link travel costs are finite, namely, for all $\ell \in \mathcal{L}$

$$\tau_{\ell}(x_{\ell}) < \infty \text{ if } x_{\ell} < \infty.$$

Assumption (A3) disregards cases where travel times are unbounded, and will be relaxed later in this section.

Next, we recall the graph-theoretic notion of min-cut capacity [21]. Let the set of nodes \mathcal{V} be partitioned into two subsets $\mathcal{S} \subseteq \mathcal{V}$ and $\bar{\mathcal{S}} = \mathcal{V} - \mathcal{S}$, such that the network source $s \in \mathcal{S}$ and the network destination $d \in \bar{\mathcal{S}}$. Let $\mathcal{S}^{\text{out}} = \{(v, u) \in \mathcal{L} : v \in \mathcal{S} \text{ and } u \in \bar{\mathcal{S}}\}$ be a cut, namely, the set of all links from \mathcal{S} to $\bar{\mathcal{S}}$, and let $C_{\mathcal{S}} = \sum_{\ell \in \mathcal{S}^{\text{out}}} C_{\ell}$ be the capacity of the cut. The min-cut capacity is defined as

$$C_{\text{m-cut}} = \min_{\mathcal{S}} C_{\mathcal{S}}.$$

The following result relates the existence of fixed points to the magnitude of the exogenous inflow to the network.

Theorem 3.2: (Existence of Equilibria) Let Assumptions (A1)-(A3) be satisfied. The interconnected system (7) admits an equilibrium point that satisfies (10) if and only if the network inflow is no larger than the min-cut capacity:

$$\bar{\lambda} \leq C_{\text{m-cut}}. \quad (11)$$

Proof: (If) The proof is organized into two main parts.

First, we show that, when $\bar{\lambda} \leq C_{\text{m-cut}}$, there exists a pair (x^*, r^*) that is an equilibrium point of the traffic dynamics with finite perceived travel costs, that is,

$$(R^{*\text{T}} - I)f(x^*) + \lambda = 0, \text{ and } \pi(x^*) < \infty.$$

To this aim, consider the graph \mathcal{G} with associated inflow $\bar{\lambda}$. By application of the max-flow min-cut theorem [21], there exists a feasible assignment of flows to the links of the graph \mathcal{G} , that is, a set of scalars $\{\varphi_1, \dots, \varphi_n\}$ such that the following conditions are satisfied:

$$\begin{aligned} 0 \leq \varphi_\ell \leq C_\ell, & \quad \text{for all } \ell \in \mathcal{L}, \\ \sum_{\ell \in v^{\text{in}}} \varphi_\ell = \sum_{\ell \in v^{\text{out}}} \varphi_\ell, & \quad \text{for all } v \in \mathcal{V}, \\ \varphi_1 = \bar{\lambda}. & \end{aligned}$$

By choosing $r_{\ell m}^* := \varphi_m / \varphi_\ell$ for all $(\ell, m) \in \mathcal{A}$, the above equations imply that

$$(R^{*\text{T}} - I)\varphi + \lambda = 0,$$

where $\varphi = [\varphi_1 \dots, \varphi_n]^\text{T}$. Finally, by choosing x_ℓ^* so that $f_\ell^{\text{out}}(x_\ell^*) = \varphi_\ell$, we have that (x^*, r^*) is a fixed point of the traffic dynamics (2), which proves the first claim.

Second, we show that for any traffic state $x^* \in \mathcal{X}$ with finite perceived costs, $\pi(x^*) < \infty$, there exists a vector of feasible routing ratios r^* that is a fixed point of the routing dynamics and satisfies (10). To this aim, we first consider a given link $\ell \in \mathcal{L}$ and we prove the claim for the single junction equation (4). The statement will then follow by iterating the reasoning for all $\ell \in \mathcal{L}$. We distinguish among two cases.

(Case 1) For all pairs $a, \bar{a} \in \mathcal{A}_\ell$, where $a = (\ell, m)$ and $\bar{a} = (\ell, \bar{m})$, the costs satisfy $\pi_m(x^*) = \pi_{\bar{m}}(x^*)$. In this case, the following identity holds:

$$\begin{aligned} \dot{r}_{\ell m}^* &= r_{\ell m}^* \left(\sum_q r_{\ell q}^* \pi_q^* - \pi_m^* \right) \\ &= r_{\ell m}^* \left(\underbrace{\pi_m^* \sum_q r_{\ell q}^*}_{=1} - \pi_m^* \right) \\ &= r_{\ell m}^* \underbrace{(\pi_m^* - \pi_m^*)}_{a_{\ell m}(x^*)} = 0, \end{aligned}$$

which shows that $\dot{r}_{\ell m}^* = 0$ and $a_{\ell m}^* = 0$, and proves that (x^*, r^*) is an equilibrium point that satisfies (10).

(Case 2) There exists $a, \bar{a} \in \mathcal{A}_\ell$, where $a = (\ell, m)$ and $\bar{a} = (\ell, \bar{m})$, such that the costs satisfy $\pi_m \neq \pi_{\bar{m}}$. In this case, let

$$\pi_{\bar{m}} = \max_{a=(\ell, m) \in \mathcal{A}_\ell} \pi_m,$$

be the largest perceived cost at the junction. By letting $r_{\ell \bar{m}}^* = 1$ and $r_{\ell m}^* = 0$ for all $m \neq \bar{m}$ we obtain the following identities

$$\begin{aligned} \dot{r}_{\ell \bar{m}}^* &= r_{\ell \bar{m}}^* \left(\sum_q r_{\ell q}^* \pi_q^* - \pi_{\bar{m}}^* \right) = (\pi_{\bar{m}}^* - \pi_{\bar{m}}^*) = 0, \\ \dot{r}_{\ell m}^* &= r_{\ell m}^* \left(\sum_q r_{\ell q}^* \pi_q^* - \pi_m^* \right) = 0, \end{aligned}$$

which shows that the provided choice of r^* is a fixed point of (4). Moreover, the above identities also imply

$$\begin{aligned} a_{\ell \bar{m}}(x^*) &= (\pi_{\bar{m}}^* - \pi_{\bar{m}}^*) = 0, \\ a_{\ell m}(x^*) &= (\pi_{\bar{m}}^* - \pi_m^*) > 0, \end{aligned}$$

which shows that the equilibrium point satisfies (10).

The conclusion thus follows by combining the two parts of the proof. In fact, when $\bar{\lambda} \leq C_{\text{m-cut}}$ the first part shows that the traffic dynamics admit an equilibrium with finite perceived costs for *some* choice of the routing. The second part of the proof guarantees that the routing dynamics admit an equilibrium for *any* traffic state with finite travel costs.

(Only if) The proof of this statement follows by adopting a contradiction reasoning. To this aim, assume (x^*, r^*) is an equilibrium point and that $\bar{\lambda} > C_{\text{m-cut}}$. The latter assumption, combined with the Maximum Flow Theorem, implies that for any assignment of flows to the links of the graph \mathcal{G} there exists $\ell \in \mathcal{L}$ such that $\varphi_\ell > C_\ell$. In other words, link ℓ is required to transfer a traffic flow $f_\ell^{\text{in}}(x^*) = \varphi_\ell$, and thus:

$$\begin{aligned} \dot{x}_\ell &= f_\ell^{\text{in}}(x^*) - f_\ell^{\text{out}}(x_\ell^*) \\ &= \varphi_\ell - f_\ell^{\text{out}}(x_\ell^*) \\ &\geq \varphi_\ell - C_\ell > 0, \end{aligned}$$

which shows that x_ℓ grows unbounded, and hence contradicts the assumption that (x^*, r^*) is an equilibrium. ■

The above theorem bridges an interesting gap between the behavior of dynamical systems and graph-theoretic notions. In fact, it relates the properties of the equilibrium points of a dynamical system with the notion of minimum-cut capacity, which is a feature of static graphs. Two important implications follow from Theorem 3.2. First, by recalling that the minimum-cut capacity equals the maximum flow through a graph (see Maximum-Flow Theorem [21]), the result shows that a dynamical traffic network admits an equilibrium point that transfers a traffic demand equal to the maximum flow. This observation demonstrates that routing apps not only optimize the travelers' commute, but also have a benefit at the system-level. Second, the result shows that when the traffic demand is too large ($\bar{\lambda} > C_{\text{m-cut}}$), then the network does not admit any equilibrium point, in fact, it operates at a condition in which traffic densities in the links grow unbounded.

We conclude this section by discussing a special technical assumption that can be used to capture back propagation of traffic congestion, a scenario that is particularly relevant in practice. To this aim, we introduce the following assumption.

(A4) For all $\ell \in \mathcal{L}$, the travel cost becomes unbounded when ℓ reaches its flow capacity, namely,

$$\tau_\ell(x_\ell) = \infty \text{ for all } x_\ell \text{ such that } f_\ell^{\text{out}}(x_\ell) = C_\ell.$$

Assumption (A4) states that if a link is approaching its maximum flow capacity, then the travelers will increasingly avoid it. This setting can also be used to capture back propagation, where if the density of a link reaches a critical value then no additional vehicles can enter that link (cf. Remark 1). The following corollary refines Theorem 3.2 for unbounded costs.

Corollary 3.3: Let Assumption (A4) replace (A3) in Theorem 3.2. The interconnected system (7) admits an equilibrium

point that satisfies (10) if and only if the network inflow is strictly lower than the min-cut capacity:

$$\bar{\lambda} < C_{\text{m-cut}}.$$

C. Relationship to Wardrop Equilibrium

The following result relates the fixed points of the dynamical system (7) with the established notion of Wardrop equilibria.

Theorem 3.4: (Relationship Between Fixed Points and Wardrop Equilibria) Consider the interconnected system (7). The following statements are equivalent:

- (i) $x^* \in \mathcal{X}$ satisfies the Wardrop First Principle;
- (ii) The pair (x^*, r^*) is a fixed point of (7) for some $r^* \in \mathcal{R}_{\mathcal{G}}$. Moreover, (x^*, r^*) satisfies (10).

Proof: (i) \Rightarrow (ii) We begin by observing that, by assumption, a Wardrop equilibrium is also an equilibrium of the traffic dynamics (2). Thus, we next prove that given a vector x^* that satisfies the Wardrop conditions, there exists a vector $r^* \in \mathcal{R}_{\mathcal{G}}$ such that (10) is satisfied.

Since the graph is acyclic, it admits a shortest path spanning tree [21], that is, a directed tree rooted from the source with the property that the unique path from the source to any node is a shortest path to that node. Notice that, since in general the Wardrop First Principle allows the existence of multiple paths with optimal travel costs, the shortest-path spanning tree is typically not unique. Next, we distinguish among three cases.

(Case 1) For all $p, \bar{p} \in \mathcal{P}$, $\tau_p^* - \tau_{\bar{p}}^* = 0$, namely, all origin-destination paths have identical travel time. This assumption implies that for every node $v \in \mathcal{V}$ all its outgoing links belong to one of the shortest-path spanning trees. This observation, combined with the fact that the perceived costs are equal to the shortest travel cost to destination, implies that

$$\pi_q^* = \pi_m^*, \text{ for all pairs } m, q \in v^{\text{out}}.$$

As a result,

$$a_{\ell m}(x^*) = \sum_q r_{\ell q}^* \pi_q^* - \pi_m^* = \pi_m^* \left(\sum_q r_{\ell q}^* - 1 \right) = 0.$$

By iterating the above equation for all $(\ell, m) \in \mathcal{A}$ we proved that the first condition in (10) is satisfied.

(Case 2) There exists a unique $p \in \mathcal{P}$ such that $f_p^* = 0$ and for all $\bar{p} \neq p$, $\tau_{\bar{p}}^* - \tau_p^* \leq 0$, namely, the path p has suboptimal travel time to destination. This assumption implies that there exists a certain node in the network $v \in \mathcal{V}$ such that one of its outgoing links $m \in v^{\text{out}}$ belongs to p (i.e., it does not belong to any shortest-path spanning tree), while $q \in v^{\text{out}}$ belongs to some \bar{p} (i.e., it belongs to a shortest path spanning tree). This observation, combined with the fact that the perceived costs are equal to the shortest travel cost to destination, implies

$$\pi_m^* > \pi_q^*.$$

Moreover, since m belongs to an origin-destination path with zero flow, there exists $\ell \in v^{\text{in}}$ such that $r_{\ell m}^* = 0$, and thus

$$\begin{aligned} a_{\ell m}(x^*) &= \sum_q r_{\ell q}^* \pi_q^* - \pi_m^* \\ &= \sum_{q \neq m} r_{\ell q}^* \pi_q^* + \underbrace{r_{\ell m}^*}_{=0} \pi_m^* - \pi_m^* \\ &= \sum_{q \neq m} r_{\ell q}^* \pi_q^* - \pi_m^* \\ &= \pi_q^* \sum_{q \neq m} r_{\ell q}^* - \pi_m^* \\ &= \pi_q^* - \pi_m^* < 0, \end{aligned} \quad (12)$$

which proves that $a_{\ell m}(x^*) < 0$, and shows that the second condition in (10) is satisfied for the pair $(\ell, m) \in \mathcal{A}$.

(Case 3) There exists multiple $p \in \mathcal{P}$ such that $f_p^* = 0$ and for some $\bar{p} \neq p$, $\tau_{\bar{p}}^* - \tau_p^* \leq 0$, namely, there exists multiple origin-destination paths with suboptimal travel cost. Under this assumption, we note that the bound derived in (12) can be iterated for all links m such that $r_{\ell m}^* = 0$, which shows that the second condition in (10) is satisfied for all these pairs, and concludes the proof of the implication.

(ii) \Rightarrow (i) To prove this implication we consider three cases.

(Case 1) For all $(\ell, m) \in \mathcal{A}$, $a_{\ell m}(x^*) = 0$, namely, all links have identically zero appeal. Under this assumption, for every $\ell \in \mathcal{L}$, all the perceived travel costs satisfy

$$0 = a_{\ell m}(x^*) = \sum_q r_{\ell q}^* \pi_q^* - \pi_m^*, \text{ for all } m \in \mathcal{A}_{\ell}, \quad (13)$$

which implies that $\pi_m^* = \pi_{\bar{m}}^*$ for all $m, \bar{m} \in \mathcal{A}_{\ell}$ are identical (i.e. $\pi_m^* = \pi_{\bar{m}}^* = \sum_q r_{\ell q}^* \pi_q^*$). This observation, combined with the fact that the perceived costs are equal to the shortest travel cost to destination, implies that every link in the network belongs to a shortest path to destination. Hence, all origin-destination paths have identical travel cost, i.e. $\tau_p^* - \tau_{\bar{p}}^* = 0$, which shows that x^* satisfies the Wardrop First Principle.

(Case 2) There exists a unique $(\ell, m) \in \mathcal{A}$ such that $a_{\ell m}(x^*) < 0$ and $r_{\ell m} = 0$. Under this assumption, we first prove that there exists a path $p \in \mathcal{P}$ containing link m such that $f_p^* = 0$. Since the flow on any origin-destination path can be written as the network inflow multiplied by the product of the routing ratios belonging to that path:

$$f_p^* = \bar{\lambda} \prod_{q, w \in p} r_{qw}^*,$$

we immediately obtain $f_p^* = 0$.

Second, we prove that for all $\bar{p} \in \mathcal{P}$, $\bar{p} \neq p$, the following inequality holds: $\tau_{\bar{p}}^* - \tau_p^* \leq 0$. By using the assumption $a_{\ell m}(x^*) < 0$, together with $\pi_w^* = \sum_q r_{\ell q}^* \pi_q^*$, which holds for all $w \in \mathcal{A}_{\ell}$, $w \neq m$, (see (13)), we have

$$\begin{aligned} 0 > a_{\ell m}(x^*) &= \sum_q r_{\ell q}^* \pi_q^* - \pi_m^* \\ &= \pi_w^* \sum_q r_{\ell q}^* - \pi_m^* \\ &= \pi_w^* - \pi_m^*. \end{aligned} \quad (14)$$

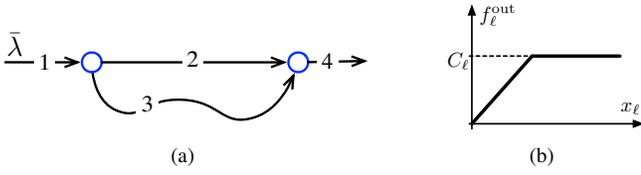


Fig. 3. Two-link network (a), and piecewise affine outflow function (b).

Since path p contains link w and the minimum travel cost from w to destination is suboptimal ($\pi_w^* > \pi_m^*$), we have that any path $\bar{p} \in \mathcal{P}$ containing link m satisfies

$$\tau_{\bar{p}}^* < \tau_p^*,$$

which shows that x^* satisfies the Wardrop First Principle.

(Case 3) There exists multiple ramps $(\ell, m) \in \mathcal{A}$ such that $a_{\ell m}(x^*) < 0$ and $r_{\ell m}^* = 0$. Under this assumption, we note that equation (14) still applies because $r_{\ell m}^* = 0$. Hence, the reasoning adopted for (Case 2) can be iterated for all (ℓ, m) such that $a_{\ell m}(x^*) < 0$ and $r_{\ell m}^* = 0$. ■

Three important implications follow from the above theorem. First, the result shows that a Wardrop equilibrium is also an equilibrium of the dynamical model (7), thus showing that if a dynamical network starts at a Wardrop equilibrium it will remain at that equilibrium at all times. Second, the result shows that dynamical systems in which travelers update their routing in real-time at every junction by minimizing their perceived travel cost admit equilibrium points that satisfy the Wardrop conditions. This observation supports our modeling choices, and demonstrates that the perceived costs are representative quantities to describe the economical decisions of routing apps. Third, by combining Theorem 3.4 with (10), it follows that a Wardrop Equilibrium is perceived by the travelers when all the network links have a nonpositive appeal function. This condition corresponds to a situation where at every junction no link is more appealing than others.

IV. STABILITY ANALYSIS

In this section, we characterize the stability of the fixed points of the feedback interconnection (7). Our main findings are summarized in the following theorem.

Theorem 4.1: (Stability of Interconnected Traffic Dynamics) Let (x^*, r^*) be a fixed point of (7) satisfying the conditions (10). Then, (x^*, r^*) is stable.

The proof of this theorem is postponed to later in this section.

The simple stability of the fixed points implies that the state trajectories are not guaranteed to decay asymptotically towards the equilibrium points, and can result in nontrivial behaviors, such as oscillations, as illustrated in the following example.

Example 4: (Existence of Oscillations in Two Parallel Roads) Consider the network illustrated in Fig. 3(a), representing two parallel roads subject to a constant inflow of vehicles $\bar{\lambda} \in \mathbb{R}_{>0}$. We assume that the travel costs are linear

$$\tau_\ell(x_\ell) = x_\ell,$$

and that all outflows are identical and piecewise-affine:

$$f_\ell^{\text{out}}(x_\ell) = \min\{vx_\ell, C\},$$

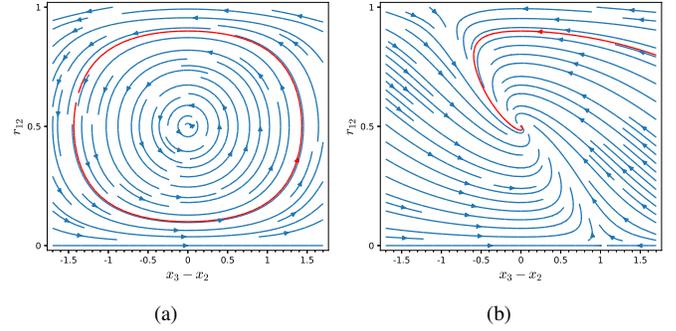


Fig. 4. Phase portrait: (a) oscillatory trajectories, (b) stable trajectories. The red curve illustrates an example of trajectories passing through the conditions $x_3 = x_2$ and $r_{12} = 0.9$.

for all $\ell \in \mathcal{L} = \{1, \dots, 4\}$, where $v \in \mathbb{R}_{>0}$.

We distinguish among two cases: (a) the network is operating in congested regimes, that is, at all times $x_1 > C/v$ and $x_2 > C/v$, and (b) the network is operating in regimes of free-flow, that is, at all times $x_1 \leq C/v$ and $x_2 \leq C/v$. Fig. 4 (a) and (b) show the phase portrait of the system trajectories in case (a) and case (b), respectively. As illustrated by the plots: in case (a) the trajectories of the system are oscillating periodic orbits; in contrast, in case (b) the trajectories converge asymptotically to an equilibrium point. The presence of periodic orbits implies that the equilibrium points are stable, but not asymptotically stable, thus supporting Theorem 4.1.

The existence of periodic orbits in case (a) can be further formalized. To this aim, we recall the dynamical equations governing the system in this regime:

$$\begin{aligned} \dot{x}_2 &= -C + r_{12}\bar{\lambda}, \\ \dot{x}_3 &= -C + r_{13}\bar{\lambda}, \\ \dot{r}_{12} &= r_{12}(1 - r_{12})(x_3 - x_2), \end{aligned}$$

where we used the fact that $f_1^{\text{out}} = \bar{\lambda}$ after an initial transient. This system admits an equilibrium point described by $r_{12} = 0.5$ and $x_3 = x_2$. We adopt the change of variables $z := x_3 - x_2$, and rewrite the dynamical equations describing the new state $[z, r_{12}]^T$:

$$\begin{aligned} \dot{z} &= (1 - 2r_{12})\bar{\lambda}, \\ \dot{r}_{12} &= r_{12}(1 - r_{12})z. \end{aligned}$$

Next, we show that the following quantity:

$$U(z, r_{12}) := \frac{1}{2}z^2 - \bar{\lambda}(\ln r_{12} - \ln(1 - r_{12})),$$

is conserved along the trajectories of the system. To this aim, we compute its time derivative to obtain

$$\begin{aligned} \dot{U}(z, r_{12}) &= z\dot{z} - \bar{\lambda}\left(\frac{1}{r_{12}} + \frac{1}{1 - r_{12}}\right)\dot{r}_{12} \\ &= z(1 - 2r_{12})\bar{\lambda} - \bar{\lambda}((1 - r_{12})z - r_{12}z) \\ &= z(1 - 2r_{12})\bar{\lambda} - z(1 - 2r_{12})\bar{\lambda} = 0, \end{aligned}$$

which shows that the quantity $U(z, r_{12})$ is a constant of motion, and proves the existence of periodic orbits. □

In the remainder of this section, we illustrate the key technical results that prove Theorem 4.1. In short, the stability of the fixed points of (7) follows from interpreting the system as a negative feedback interconnection between the traffic dynamics and the routing dynamics (see Fig. 2), and from showing that each open-loop component is a passive dynamical system. We refer to Appendix A-B for a summary of the notions of passivity utilized in the remainder.

We next show that the routing dynamics satisfy the passivity property. To this aim, we first prove that the group of routing equations at a single junction are passive. For every link $\ell \in \mathcal{L}$, recall that \mathcal{A}_ℓ is the set of links available at the downstream junction, and let $|\mathcal{A}_\ell| := \alpha$ be its cardinality. We interpret the set of α dynamical equations

$$\dot{r}_{\ell m} = r_{\ell m} \left(\sum_q r_{\ell q} \pi_q - \pi_m \right), \text{ for all } m \in \mathcal{A}_\ell, \quad (15)$$

as a dynamical system with input and output, respectively,

$$\begin{aligned} u_\ell &= [\pi_{m_1}, \dots, \pi_{m_\alpha}]^\top, \\ y_\ell &= [r_{\ell m_1}, \dots, r_{\ell m_\alpha}]^\top. \end{aligned} \quad (16)$$

The following result formalizes the passivity of equations (15).

Lemma 4.2: (Passivity of Single-Junction Routing Dynamics) The single-junction routing dynamics (15) is passive with respect to the input-output pair $(-u_\ell, y_\ell)$.

Proof: We let $[r_{\ell m_1}^*, \dots, r_{\ell m_p}^*]$ denote a fixed point of (15), and we show that

$$V_\ell(r) = \sum_{m \in \mathcal{A}_\ell} r_{\ell m}^* \ln \left(\frac{r_{\ell m}^*}{r_{\ell m}} \right), \quad (17)$$

is a storage function for the dynamical system defined by (15). We begin by observing that V_ℓ is differentiable because it is a linear combination of natural logarithm functions. Moreover, by using the log-sum inequality, we have

$$\begin{aligned} V_\ell(r) &= \sum_m r_{\ell m}^* \ln \left(\frac{r_{\ell m}^*}{r_{\ell m}} \right) \\ &\geq \sum_m r_{\ell m}^* \ln \left(\frac{\sum_m r_{\ell m}^*}{\sum_m r_{\ell m}} \right) \\ &= \ln(1) = 0, \end{aligned}$$

where we used the fact that $\sum_m r_{\ell m}^* = \sum_m r_{\ell m} = 1$, which shows that V_ℓ is an appropriate choice of storage function.

To show the passivity property, we first incorporate the negative sign of the input vector into the dynamical equation, and we rewrite (15) as

$$\dot{r}_{\ell m} = r_{\ell m} \left(\pi_m - \sum_q r_{\ell q} \pi_q \right), \text{ for all } m \in \mathcal{A}_\ell,$$

and we next show passivity of the above equation with respect to the input-output pair (u_ℓ, y_ℓ) . The derivative of the storage

function is

$$\begin{aligned} \dot{V}_\ell(r) &= - \sum_m r_{\ell m}^* \frac{\dot{r}_{\ell m}}{r_{\ell m}} = - \sum_m r_{\ell m}^* (\pi_m - \sum_q r_{\ell q} \pi_q) \\ &= - \sum_m r_{\ell m}^* \pi_m + \underbrace{\sum_m r_{\ell m}^* \sum_q r_{\ell q} \pi_q}_{=1} \\ &= - \sum_m r_{\ell m}^* \pi_m + \sum_q r_{\ell q} \pi_q \\ &\leq \sum_q r_{\ell q} \pi_q = u_\ell^\top y_\ell, \end{aligned}$$

where for the last inequality we used the fact that $r_{\ell q} \geq 0$ and $\pi_q \geq 0$, which shows the claim and concludes the proof. ■

Next, we leverage the above lemma to show that the overall routing dynamics (5) also satisfy the passivity property. To this aim, we consider (5) as a dynamical system with input and output vectors, respectively,

$$\begin{aligned} u_r &= [u_{\ell_1}, \dots, u_{\ell_n}]^\top, \\ y_r &= [y_{\ell_1}, \dots, y_{\ell_n}]^\top, \end{aligned} \quad (18)$$

where u_{ℓ_i} and y_{ℓ_i} , $i \in \{1, \dots, n\}$, are defined in (16). Passivity of the overall routing dynamics is formalized next.

Lemma 4.3: (Passivity of Overall Routing Dynamics) Let the perceived travel costs be modeled as in (3). Then, the overall routing dynamics (5) is passive with respect to the input-output pair $(-u_r, y_r)$.

Proof: The proof of this statement consists of two parts. First, we show that the dynamical equations at every pair of junctions are independent, and thus the overall routing dynamics (5) can be studied as a composition of independent subsystems. To this aim, we will show that u_ℓ is independent of y_m in (16), for all $\ell \neq m$. This fact immediately follows by observing that, when the perceived travel costs follow the model (3), the perceived cost $\pi_\ell(x)$ is a function that only depends on x , and it is independent of the routing r .

Second, we show that passivity of all the individual junctions implies passivity of the overall routing dynamics (5). To this aim, we consider the following storage function for (5):

$$V_r(r) = \sum_{\ell \in \mathcal{L}} V_\ell(r), \quad (19)$$

where V_ℓ denotes the storage function associated to junction ℓ . By taking the time derivative of the above storage function:

$$\dot{V}_r(r) = \sum_{\ell \in \mathcal{L}} \dot{V}_\ell(r) \leq \sum_{\ell \in \mathcal{L}} u_\ell^\top y_\ell = u_r^\top y_r,$$

where the inequality follows from the passivity of the individual junctions, which proves the passivity of (5). ■

Next, we show that the traffic dynamics (2) satisfy the passivity property. To this aim, we interpret (2) as an input-output dynamical system with input described by the set of routing ratios, and output described by the set of perceived link costs. Formally, to the scalar input $r_{\ell m}$ we associate the scalar output π_m or, equivalently, in vector form we consider the following input and output vectors:

$$\begin{aligned} u_x &= [r_{11}, r_{12}, \dots, r_{1n}, r_{21}, \dots, r_{nn}]^\top, \\ y_x &= [\pi_1, \pi_2, \dots, \pi_n, \pi_1, \dots, \pi_n]. \end{aligned} \quad (20)$$

The following result formalizes the passivity of (2).

Lemma 4.4: (Passivity of the Traffic Dynamics) Assume that all links $\ell \in \mathcal{L}$ have finite flow capacity $C_\ell < \infty$. Then, the traffic network (2) is a passive dynamical system with respect to the input-output pair (u_x, y_x) .

Moreover, if for all ℓ there exists $\rho_\ell \in \mathbb{R}_{>0}$ such that

$$f_\ell(x_\ell) \geq \rho_\ell \pi_\ell(x_\ell), \quad (21)$$

then the traffic dynamics (2) are output strictly passive.

Proof: We show that the following function

$$V_x(x) = \frac{1}{h} \sum_{\ell \in \mathcal{L}} \int_0^{x_\ell} \pi_\ell(\sigma) d\sigma, \quad (22)$$

is a storage function for (2), where the constant $h \in \mathbb{R}_{>0}$ is chosen as follows:

$$h = \max_{\ell \in \mathcal{L}} C_\ell.$$

We note that V_x is non-negative and it is differentiable, because it is the combination of integral functions, and thus it is an appropriate choice of storage function. By taking the time derivative of the storage function we obtain

$$\begin{aligned} \dot{V}_x(x) &= \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \dot{x}_\ell \\ &= \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \left(-f_\ell^{\text{out}}(x_\ell) + \sum_{m \in \mathcal{A}_\ell} r_{m\ell} f_m^{\text{out}}(x_m) \right) \\ &= -\frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) f_\ell^{\text{out}}(x_\ell) \\ &\quad + \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \sum_{m \in \mathcal{A}_\ell} r_{m\ell} f_m^{\text{out}}(x_m) \\ &\leq \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \sum_{m \in \mathcal{A}_\ell} r_{m\ell} f_m^{\text{out}}(x_m) \\ &\leq \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{A}_\ell} \pi_\ell(x_\ell) r_{m\ell} = u_x^\top y_x, \end{aligned}$$

where for the first inequality we used the fact that $\pi_\ell(x_\ell) f_\ell(x_\ell) \geq 0$ for all $\ell \in \mathcal{L}$, and the last inequality follows from the above choice of h (which implies $f_m/h < 1$, for all $m \in \mathcal{L}$). Hence, the bound proves the passivity of (2).

To show output-strict passivity, we substitute the inequality (21) into the time-derivative of the storage function to obtain:

$$\begin{aligned} \dot{V}_x(x) &= -\frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) f_\ell^{\text{out}}(x_\ell) \\ &\quad + \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \sum_{m \in \mathcal{A}_\ell} r_{m\ell} f_m^{\text{out}}(x_m) \\ &\leq -\frac{1}{h} \sum_{\ell \in \mathcal{L}} \rho_\ell \pi_\ell(x_\ell)^2 + \frac{1}{h} \sum_{\ell \in \mathcal{L}} \pi_\ell(x_\ell) \sum_{m \in \mathcal{A}_\ell} r_{m\ell} f_m^{\text{out}}(x_m) \\ &\leq -\sum_{\ell \in \mathcal{L}} \rho_\ell \pi_\ell(x_\ell)^2 + \sum_{\ell \in \mathcal{L}} \sum_{m \in \mathcal{A}_\ell} \pi_\ell(x_\ell) r_{m\ell} \\ &\leq -y_x^\top y_x + u_x^\top y_x, \end{aligned}$$

where the last inequality follows from $\rho_\ell > 0$, which shows that (2) is output strictly passive and concludes the proof. ■

The additional assumption (21) needed to ensure output strict passivity can be interpreted as follows. By using the fact that for every $x_\ell \in \mathcal{X}$, $\pi_\ell(x_\ell) > 0$, the inequality (21) can be rewritten as follows:

$$\frac{f_\ell(x_\ell)}{\pi_\ell(x_\ell)} \geq \rho_\ell,$$

and, by deriving both quantities with respect to x_ℓ we obtain

$$\frac{df_\ell(x_\ell)}{d\pi_\ell(x_\ell)} > 0,$$

where we used the fact that $\rho_\ell > 0$. Hence, in order to achieve output strict passivity, an increase in travel cost of a link must imply an increase in traffic flow in the link.

We are now ready to formally prove Theorem 4.1.

Proof of Theorem 4.1: To prove the stability of the fixed points, we interpret (7) as a negative feedback interconnection between the traffic dynamics and the routing dynamics, and we leverage the Passivity Theorem [15] to infer the Lyapunov stability of the system. We refer to Theorem A.1 in Appendix A-B for a concise statement of the Passivity Theorem.

We begin by observing that the lemmas 4.3 and 4.4 ensure passivity of the open loop systems. Next, we show that the equilibrium points are local minima for the storage function of routing (19) and for the storage function of traffic (22). First, we observe that the routing storage function $V_r(r)$ in (19) is the summation of the storage functions at the junctions (17), which are non-negative quantities that are identically zero at the equilibrium points $V_\ell(r^*) = 0$. Hence, the equilibrium points are local minima of the function $V_r(r)$.

Second, we show that $V_x(x)$ attains a minimum at the equilibrium points. To this aim, we first let $\bar{\lambda} = 0$ and we study the equilibrium points of (2). Every equilibrium point x^* satisfies the following identity

$$0 = (R^\top - I)f(x^*).$$

By observing that $(R^\top - I)$ is invertible (see e.g. [17, Theorem 1]), and that $f(x^*) = 0$ only if $x^* = 0$ (see Assumption (A1)), the above equation implies that the unique equilibrium point of the system satisfies $x^* = 0$. The choice of $V_x(x)$ in (22) implies that $V_x(x)$ is non-negative and that $V_x(x^*) = 0$, which shows that x^* is a local minima of the storage function. Lastly, we observe that any nonzero $\bar{\lambda}$ has the effect of shifting the equilibrium point, and thus it does not change the properties of the storage function.

Finally, the stability of the equilibrium points follows by application of condition (i) in Theorem A.1. ■

V. ROBUST INFORMATION DESIGN

In this section, we propose a control technique to guarantee the asymptotic stability the equilibrium points, and thus strengthen the robustness of the system. The method relies on regulating the rate at which travelers react to congestion by properly modifying the reaction rates. To this aim, we next introduce the notion of congestion-aware reaction rates.

Definition 2: (Congestion-Aware Reaction Rates) A set of reaction rates for the dynamics (5) is *congestion-aware* if, for all $\ell \in \mathcal{L}$,

$$\delta_\ell : \mathcal{T}^n \rightarrow (0, +\infty).$$

Moreover, let $\mathcal{T}_\ell := \{\pi_m\}_{m \in \mathcal{A}_\ell} \subseteq \mathcal{T}^n$ be the set of perceived costs at the links downstream of ℓ . A set of congestion-aware reaction rates is *local* if, for all $\ell \in \mathcal{L}$,

$$\delta_\ell : \mathcal{T}_\ell \rightarrow (0, +\infty). \quad (23)$$

□

The class of congestion-aware reaction rates conceptualizes a setting where the rate at which travelers react to changes in traffic congestion is a function of the instantaneous perceived costs. We observe that this control scheme can be achieved, for instance, by appropriately designing the rate at which navigation apps update their routing suggestions. Similarly, local congestion-aware reaction rates model a setting where, at every node of the network, the reaction rates at the incoming links only depend on the perceived costs at the outgoing links.

We observe that the adoption of congestion-aware reactions does not alter the equilibrium points of the interconnected system (7). In fact, reaction rates are positive multiplicative quantities in the dynamical equation (4), and thus the properties of the equilibrium points discussed in Section III remain unchanged. The following result characterizes the stability of the equilibrium points under congestion-aware reaction rates.

Theorem 5.1: (Asymptotic Stability Under Congestion-Aware Reaction Rates) Consider the interconnected system (7) where the routing dynamics adopt the class of local congestion-aware reaction rates (23). Moreover, assume that for every link $\ell \in \mathcal{L}$ there exists a scalar $\rho_\ell \in \mathbb{R}_{>0}$ such that

$$f_\ell(x_\ell) \geq \rho_\ell \pi_\ell(x_\ell).$$

Then, every equilibrium point (x^*, r^*) that satisfies (10) is asymptotically stable.

The proof of this theorem is postponed to later in this section.

The above theorem shows that the class of local congestion-aware reaction rates ensures the asymptotic stability of the equilibrium points of the interconnected traffic-routing system.

The result has an important practical interpretation that we illustrate next. Let $p = (\{v_1, \dots, v_k\}, \{\ell_1, \dots, \ell_k\})$ be an origin-destination path in the graph, that is, $\ell_1 = s$ and $\ell_k = d$. It follows from the recursive definition of perceived costs (3) that the perceived costs are non-increasing along a path, that is, for all $i \in \{1, \dots, n-1\}$,

$$\pi_{\ell_i} \geq \pi_{\ell_{i+1}}.$$

By combining this observation with the definition of local reactions (23), it follows that the magnitude of the reaction rates is non-increasing along a path, that is,

$$\delta_{\ell_i} \geq \delta_{\ell_{i+1}}.$$

Hence, Theorem 5.1 states that in order to achieve asymptotic stability of the equilibrium points, travelers that are closer to the network origin must react faster to changes in traffic congestion as compared to travelers that are in the proximity of the network destination.

In the remainder of this section, we present the key technical results that formally prove Theorem 5.1. Loosely speaking, the asymptotic stability of the equilibrium points follows by ensuring that the open loop components of the negative feedback interconnection (7) satisfy a strong passivity notion. We begin by proving this property for the routing dynamics.

Lemma 5.2: (Input Strict Passivity of Routing Dynamics) Assume the routing dynamics (5) adopt the class of local congestion-aware reaction rates (23). Then, the overall routing dynamics (5) is input strictly passive.

Proof: The proof of this statement consists of two parts. First, we show that the single-junction routing dynamics (15) are input strictly passive. To this aim, similarly to the proof of Lemma 4.2, we reverse the sign of the dynamical equation to take into account the negative sign in the input. Moreover, we consider the following storage function

$$V_\ell(r) = \frac{1}{h} \sum_{m \in \mathcal{A}_\ell} r_{\ell m}^* \ln \left(\frac{r_{\ell m}^*}{r_{\ell m}} \right),$$

where $[r_{\ell m_1}^*, \dots, r_{\ell m_p}^*]$ is a fixed point of (15), and the scalar $h_\ell > 0$ is chosen so that

$$h = \max_{\ell \in \mathcal{L}} \delta_\ell.$$

We observe (23) implies $h < \infty$, and that $V_\ell(r)$ is an appropriate choice of storage function (see proof of Lemma 4.2). By computing the time derivative of the storage we obtain

$$\begin{aligned} \dot{V}_\ell(r) &= -\frac{1}{h} \sum_m r_{\ell m}^* \frac{\dot{r}_{\ell m}}{r_{\ell m}} = -\frac{1}{h} \sum_m r_{\ell m}^* (\pi_m - \sum_q r_{\ell q} \pi_q) \delta_\ell \\ &= -\frac{1}{h} \sum_m r_{\ell m}^* \pi_m \delta_\ell + \frac{1}{h} \sum_m r_{\ell m}^* \underbrace{\sum_q r_{\ell q} \pi_q}_{=1} \delta_\ell \\ &= -\frac{1}{h} \sum_m r_{\ell m}^* \pi_m \delta_\ell + \frac{1}{h} \sum_q r_{\ell q} \pi_q \delta_\ell \\ &= -\frac{1}{h} \sum_m r_{\ell m}^* \pi_m \tilde{\delta}_m(\pi_m) + \frac{1}{h} \sum_q r_{\ell q} \pi_q \delta_\ell \\ &\leq -\sum_m \pi_m \tilde{\delta}_m(\pi_m) + \sum_q r_{\ell q} \pi_q \\ &= -u_\ell^\top \varphi(u_\ell) + u_\ell^\top y_\ell, \end{aligned}$$

where for the fourth identity we used the fact that δ_ℓ is a function of π_m , namely, $\delta_\ell = \tilde{\delta}_m(\pi_m)$, the inequality follows from our choice of h (which implies $\delta_\ell/h < 1$), and $\varphi(u_\ell) = [\tilde{\delta}_{m_1}(\pi_{m_1}), \dots, \tilde{\delta}_{m_p}(\pi_{m_p})]^\top$. The above inequality shows that the single-junction routing (15) is input strictly passive.

Finally, input strict passivity of the overall routing dynamics follows by combining input strict passivity of the junction dynamics with the choice of storage function $V_r(r) = \sum_{\ell \in \mathcal{L}} V_\ell(r)$ for (5), and by adopting a reasoning similar to the one used in the proof of Lemma 4.3. ■

We are now ready to formally prove Theorem 5.1.

Proof of Theorem 5.1: To prove the asymptotic stability of the fixed points, interpret (7) as a negative feedback interconnection between the traffic dynamics and the routing dynamics, and we leverage the stronger version of the Passivity

Theorem [15] for Lyapunov stability. We refer to condition (ii) of Theorem A.1 in Appendix A-B for a concise description of the assumptions required to prove this result.

The proof of this claim is organized into three main parts, and leverages the fact that the open loop components are passive dynamical systems with storage functions defined in the proofs of lemmas 4.4 and 5.2. First, we observe that Theorem 4.1 immediately implies that the equilibrium points are local minima for the storage functions.

Second, we show that each open loop systems is zero-state detectable [15]. Zero-state detectability of the routing dynamics immediately follows from the choice of input and output (16), and by observing that the state of the system coincides with its output. Zero-state detectability of the traffic dynamics immediately follows from the choice of input and output (20), and by observing that if the output of the system is identically zero then its state is identically zero.

Third, we show that the inequalities (25) are satisfied. We begin by observing that input strict passivity of the routing dynamics, proved in Lemma 5.2, ensures the existence of a function $\varphi_{\text{routing}} : \mathcal{T} \rightarrow \mathbb{R}_{>0}^n$, such that

$$v^T \varphi_{\text{routing}}(v) > 0, \text{ for all } v \neq 0.$$

Moreover, output strict passivity of the traffic dynamics, proved in Lemma 4.4, ensures the existence of a function $\rho_{\text{traffic}} : \mathcal{T} \rightarrow \mathbb{R}_{>0}^n$, such that

$$v^T \rho_{\text{traffic}}(v) > 0, \text{ for all } v \neq 0.$$

Finally, the statement of the result follows by combining the above observations, and by application of condition (ii) in the passivity theorem A.1. ■

VI. SIMULATION RESULTS

This section presents two sets of numerical simulations that illustrate our findings.

A. Data From SR60-W and I10-W in Southern California

Consider the traffic network in Fig. 5(a), which schematizes the west bounds of the freeways SR60-W and I10-W in Southern California. Let x_{60} and x_{10} be the average traffic density in the examined sections of SR60-W (absolute miles 13.1 – 22.4) and in the section of I10-W (absolute miles 24.4 – 36.02), respectively. Moreover, let r_{60} (resp. $r_{10} = 1 - r_{60}$) be the fraction of travelers choosing freeway SR60-W over I10-W (resp. choosing freeway I10-W over SR60-W) for their commute. Fig. 5(b) illustrates the time-evolution of the recorded traffic density¹ for the two freeways on Friday, March 6, 2020. The figure also illustrates an estimation of the densities and of the routing fraction as predicted by our models, demonstrating that our dynamical framework can predict the complex dynamical behaviors observed in practice.

¹Source: Caltrans Freeway Performance Measurement System (PeMS).

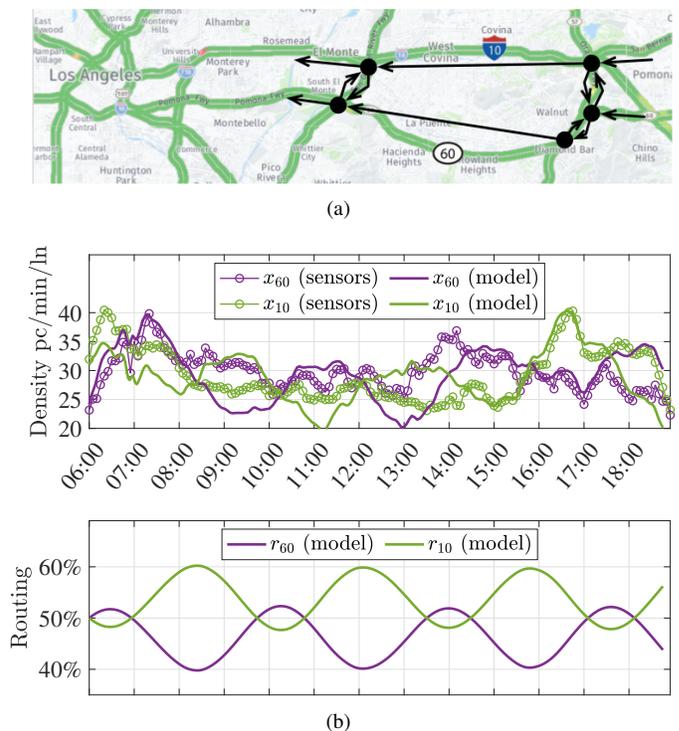


Fig. 5. Time series data for SR60-W and I10-W on March 6, 2020. (a) schematic of traffic network. (b) Sensory data (continuous lines with circles) and trajectories predicted by our models (continuous lines). (c) Routing predicted by our models. Simulation uses constant inflow $\lambda = 3340$ veh/hr/ln.

B. Oscillating Trajectories in Seven-Link Network

Consider the seven-link network discussed in Example 2 and reported in Fig. 6(a), assume that the outflows are linear

$$f_\ell(x_i) = x_i, \text{ for all } i \in \{1, \dots, 7\},$$

and that the travel costs are affine:

$$\tau_\ell(x_\ell) = a_\ell x_\ell + b_\ell, \text{ for all } i \in \{1, \dots, 7\},$$

where the parameters a_ℓ and b_ℓ are summarized in Table I. Since the flow capacities of the links are unbounded, Theorem 3.2 ensures the existence of an equilibrium point (x^*, r^*) . It can be verified that an equilibrium point that satisfies (10) is:

$$\begin{aligned} x_1^* &= 6, & x_2^* &= 4, & x_3^* &= 2, & x_4^* &= 2, & x_5^* &= 2, & x_6^* &= 4, & x_7^* &= 6, \\ r_{12} &= 2/3, & r_{13} &= 1/3, & r_{24} &= 1/2, & r_{25} &= 1/2. \end{aligned}$$

Fig. 6 shows an example of trajectories, demonstrating that the system admits a periodic orbit, which prevents the state from converging to the equilibrium points asymptotically.

Fig. 7 shows the trajectories of the same network when the routing apps use a set of local congestion-aware reaction rates, demonstrating that this class of control policies ensures the asymptotic stability of the equilibrium points.

VII. CONCLUSION

This paper proposed a dynamical routing model to understand the impact of app-informed travelers in traffic networks. We studied the stability of the routing model coupled with a dynamical traffic model, and we showed that the general

TABLE I
CHOICE OF AFFINE TRAVEL COSTS

ℓ	1	2	3	4	5	6	7
a_ℓ	1	10	1	1	1	10	1
b_ℓ	0	0	50	10	50	0	0

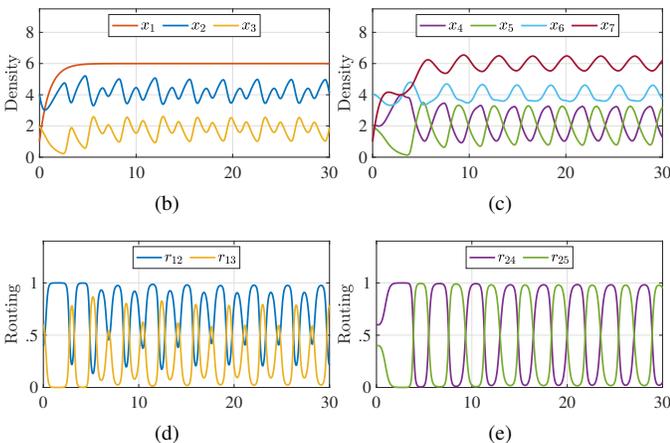
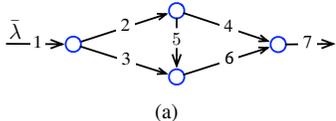


Fig. 6. (a) Seven-link network. (b)-(c) Oscillating traffic state. (d)-(e) Oscillating routing state.

adoption of routing apps (i) maximizes the throughput of flow across the traffic system, but (ii) can deteriorate the stability of the equilibrium points. To ensure asymptotic stability, we propose a control technique that relies on regulating the rate at which routing apps react to changes in traffic congestion. Our results give rise to several opportunities for future work. By coupling these models with common infrastructure-control models (such as variable speed limits and freeway metering), these results may play an important role in designing dynamical controllers for congested infrastructures. Furthermore, our models and stability analysis represent a fundamental framework for future studies on robustness and security analysis.

APPENDIX A

A. Fixed Points and Stability of Systems With Inputs

In this brief section, we gather some basic concepts on equilibria and stability of nonlinear dynamical systems.

Definition 3: (Fixed Points and Lyapunov Stability)

- A pair (x^*, r^*) and an input $\lambda^* > 0$ are a *fixed point* (or equilibrium point) of the dynamics (7) if, for initial conditions $x(0) = x^*$, $r(0) = r^*$, and constant input $\bar{\lambda} = \lambda^*$,

$$(R - I)f(x^*) + \lambda = 0, \text{ and } \varrho(r^*, \pi(x^*)) = 0.$$

- A fixed point is *stable* if, for every $\epsilon_x > 0, \epsilon_r > 0$, there exists $\delta_x > 0, \delta_r > 0$, such that

$$\begin{aligned} \|x(0) - x^*\| < \delta_x \text{ and } \|r(0) - r^*\| < \delta_r \\ \Rightarrow \|x(t) - x^*\| < \epsilon_x \text{ and } \|r(t) - r^*\| < \epsilon_r, \end{aligned}$$

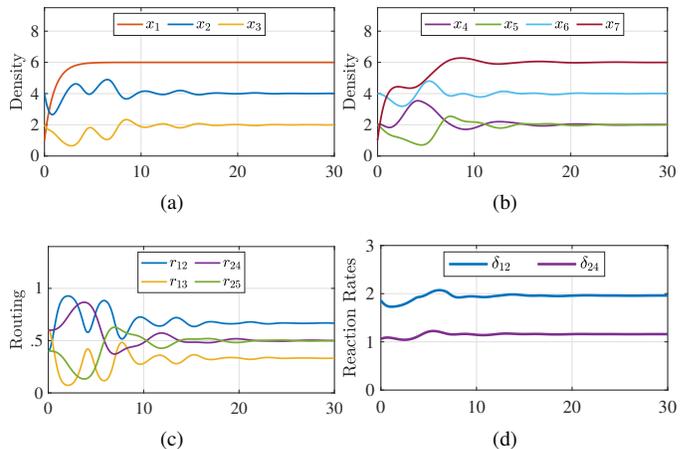


Fig. 7. Asymptotic stability under congestion-aware reaction rates.

for all $t \geq 0$.

- A fixed point is *asymptotically stable* if it is stable and

$$\lim_{t \rightarrow +\infty} x(t) = x^* \text{ and } \lim_{t \rightarrow +\infty} r(t) = r^*.$$

- A fixed point is *unstable* if it is not stable. \square

B. Passivity of Nonlinear Systems

In this section, we recall basic definitions and results on passive nonlinear dynamical systems that are instrumental for the analysis presented in this paper. We begin by recalling the definition of passivity.

Definition 4: (Passive System [15])

- A dynamical system $\dot{x} = f(x, u)$, $y = g(x, u)$, $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^p$, is *passive with respect to the input-output pair* (u, y) if there exists a differentiable function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, called the storage function, such that for all initial conditions $x(0) = x_0 \in \mathcal{X}$, for all allowed input functions $u \in \mathcal{U}$, and $t \geq 0$, the following inequality holds

$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\top y(\sigma) d\sigma. \quad (24)$$

- A dynamical system is *input strictly passive* if there exists a function $\varphi : \mathcal{U} \rightarrow \mathbb{R}_{> 0}^m$ such that $u^\top \varphi(u) > 0$ for all $u \neq 0$ and

$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\top y(\sigma) - u(\sigma)^\top \varphi(u(\sigma)) d\sigma.$$

- A dynamical system is *output strictly passive* if there exists a function $\rho : \mathcal{Y} \rightarrow \mathbb{R}_{> 0}^p$ such that $y^\top \rho(y) > 0$ for all $y \neq 0$ and

$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\top y(\sigma) - y(\sigma)^\top \rho(y(\sigma)) d\sigma.$$

Loosely speaking, a system is passive if the increase in its storage function in the time interval $[0, t]$ (left hand side of (24)) is no larger than the energy supplied to the system during that interval (right hand side of (24)). Passivity is a useful tool \square

to assess the Lyapunov stability of a feedback interconnection. The Passivity Theorem [15, Proposition 4.3.1], [22, Theorem 2.30] is summarized next.

Theorem A.1: (Passivity Theorem) Consider the systems

$$\begin{aligned} \dot{x}_i &= f(x_i, u_i), \\ y_i &= g_i(x_i, u_i), \quad i \in \{1, 2\}, \end{aligned}$$

where $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^n$, $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^m$, $y_i \in \mathcal{Y}_i \subseteq \mathbb{R}^m$, and assume the two systems are coupled by means of a negative feedback interconnection, namely $u_2 = y_1$, $u_1 = -y_2$. Moreover, assume each system is passive with storage functions $V_i : \mathcal{X}_i \rightarrow \mathbb{R}_{\geq 0}$. Then,

- (i) if V_1, V_2 have strict local minimum at x_1^*, x_2^* , then (x_1^*, x_2^*) is a stable fixed point of the negative feedback interconnection.
- (ii) Assume V_1, V_2 have strict local minimum at x_1^*, x_2^* . Moreover, assume each system is zero-state detectable and input-strictly passive or output strictly passive, namely, each system admits a storage function that satisfies

$$\begin{aligned} V_i(x(t)) - V_i(x_0) &\leq \int_0^t u_i(\sigma)^\top y_i(\sigma) - \\ &\quad u_i(\sigma)^\top \varphi_i(u_i(\sigma)) - y_i(\sigma)^\top \rho_i(y_i(\sigma)) \, d\sigma. \end{aligned}$$

for (possibly zero) functions $\varphi_i : \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}^m$ and $\rho_i : \mathcal{U}_i \rightarrow \mathbb{R}_{\geq 0}^m$. If

$$\begin{aligned} v^\top \varphi_1(v) + v^\top \rho_2(v) &> 0, \text{ and} \\ v^\top \varphi_2(v) + v^\top \rho_1(v) &> 0, \end{aligned} \quad (25)$$

for all $v \neq 0$, then (x_1^*, x_2^*) is an asymptotically stable fixed point of the negative feedback interconnection.

C. Proofs

Proof of Lemma 2.2: We equivalently show that the linear map $f^* = E f_p^*$ is injective, that is, $\text{Ker}(E) = \emptyset$. By using the fact that E is a $n \times \zeta$ matrix, we will show that the columns of E are linearly independent, that is, $\text{Rank}(E) = \zeta$. We organize the proof into two parts. First, we show that $\zeta \leq n$. Second, we prove that E contains ζ linearly independent columns.

To show that $\zeta \leq n$, we let $\mathcal{S} \subseteq \mathcal{V}$ and we partition \mathcal{V} into two subsets, \mathcal{S} and $\bar{\mathcal{S}} = \mathcal{V} - \mathcal{S}$, such that $s \in \mathcal{S}$ and $d \in \bar{\mathcal{S}}$. Moreover, we let $\mathcal{S}^{\text{out}} = \{(v_i, v_j) \in \mathcal{L} : v_j \in \mathcal{S} \text{ and } v_i \in \bar{\mathcal{S}}\}$ be the set of all links from \mathcal{S} to $\bar{\mathcal{S}}$. By application of the Max-Flow Theorem [21], the number of simple paths ζ satisfies

$$\zeta = \min_{\mathcal{S} \subseteq \mathcal{V}} |\mathcal{S}^{\text{out}}|,$$

where $|\mathcal{S}^{\text{out}}|$ denotes the cardinality of the set \mathcal{S}^{out} . Since $\mathcal{S}^{\text{out}} \subseteq \mathcal{L}$, we have $|\mathcal{S}^{\text{out}}| \leq n$ and thus $\zeta \leq n$, which concludes the first part of the proof.

To show that the columns of E are linearly independent, we denote by E_i the i -th column of E , $i \in \{1, \dots, \zeta\}$, and we observe E_i can be written as a linear combination of the form

$$E_i = \sum_{j=1}^n b_{ij} e_j,$$

where e_j denotes the j -th canonical vector of dimension n , and $b_{ij} \in \{0, 1\}$, with $b_{ij} = 1$ only if link i belongs to path p_j . To conclude, we use the fact that the graph contains ζ edge-disjoint paths, namely, each pair of paths differ by at least one edge, which shows that E contains ζ linearly-independent columns and concludes the proof. ■

REFERENCES

- [1] M. Sprung, M. Chambers, and S. Smith-Pickel, "Transportation statistics annual report 2018," U.S. Department of Transportation, Tech. Rep., 2018. [Online]. Available: <https://rosap.ntl.bts.gov/view/dot/37861>
- [2] C. F. Daganzo, "The cell transmission model part II: network traffic," *Transp. Research Part B: Methodological*, vol. 29, no. 2, pp. 79–93, 1995.
- [3] S. Coogan and M. Arcak, "A compartmental model for traffic networks and its dynamical behavior," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2698–2703, 2015.
- [4] E. Lovisari, G. Como, and K. Savla, "Stability of monotone dynamical flow networks," in *IEEE Conf. on Decision and Control*, Dec. 2014, pp. 2384–2389.
- [5] M. Patriksson, *The traffic assignment problem: models and methods*. Mineola, NY: Dover Publications, 2015.
- [6] N. Mehr and R. Horowitz, "How will the presence of autonomous vehicles affect the equilibrium state of traffic networks?" *IEEE Transactions on Control of Network Systems*, pp. 1–10, 2019, in press.
- [7] D. A. Lazar, S. Coogan, and R. Pedarsani, "The price of anarchy for transportation networks with mixed autonomy," in *American Control Conference*, June 2018, pp. 6359–6365.
- [8] J. G. Wardrop, "Some theoretical aspects of road traffic research," *Proceedings of the institution of civil engineers*, vol. 1, no. 3, pp. 325–362, 1952.
- [9] J. W. Weibull, *Evolutionary game theory*. MIT press, 1997.
- [10] S. Fischer and B. Vöcking, "On the evolution of selfish routing," in *Algorithms - ESA 2004*. Berlin, Heidelberg: Springer, 2004, pp. 323–334.
- [11] W. Krichene, B. Drighès, and A. M. Bayen, "Online learning of nash equilibria in congestion games," *SIAM Journal on Control and Optimization*, vol. 53, no. 2, pp. 1056–1081, 2015.
- [12] G. Como, K. Savla, D. Acemoglu, M. A. Dahleh, and E. Frazzoli, "Stability analysis of transportation networks with multiscale driver decisions," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 230–252, 2013.
- [13] —, "Robust distributed routing in dynamical networks - part I: Locally responsive policies and weak resilience," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 317–332, 2013.
- [14] H. K. Lo and W. Y. Szeto, "Modeling advanced traveler information services: static versus dynamic paradigms," *Transp. Research Part B: Methodological*, vol. 38, no. 6, pp. 495–515, 2004.
- [15] A. V. D. Schaft, *L2-gain and passivity techniques in nonlinear control*. Springer, 2000, vol. 2.
- [16] S. Chen, H. Yu, and M. Krstic, "Regulator design for a congested continuum traffic model with app-routing instability," *Arxiv preprint arXiv:1911.02713*, 2019.
- [17] G. Bianchin and F. Pasqualetti, "Gramian-based optimization for the analysis and control of traffic networks," *IEEE Transactions on Intelligent Transportation Systems*, pp. 1–12, 2019.
- [18] K. H. Schlag, "Why imitate, and if so, how?: A boundedly rational approach to multi-armed bandits," *Journal of Economic Theory*, vol. 78, no. 1, pp. 130 – 156, 1998.
- [19] E. Hopkins, "A note on best response dynamics," *Games and Economic Behavior*, vol. 29, no. 1, pp. 138 – 150, 1999.
- [20] W. H. Sandholm, "Potential games with continuous player sets," *Journal of Economic Theory*, vol. 97, no. 1, pp. 81 – 108, 2001.
- [21] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows*. New Jersey: Prentice Hall, 1988.
- [22] R. Sepulchre, M. Jankovic, and P. Kokotovic, *Constructive nonlinear control*. London: Springer-Verlag London, 2012.