2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

The Observability Radius of Network Systems

Gianluca Bianchin, Paolo Frasca, Andrea Gasparri, and Fabio Pasqualetti

Abstract— **This paper introduces the observability radius of network systems, which measures the robustness of a network to perturbations of the edges. We consider linear networks, where the dynamics are described by a weighted adjacency matrix, and dedicated sensors are positioned at a subset of nodes. We allow for perturbations of certain edge weights, with the objective of preventing observability of some modes of the network dynamics. Our work considers perturbations with a desired sparsity structure, thus extending the classic literature on the controllability and observability radius of linear systems. We propose an optimization framework to determine a perturbation with smallest Frobenius norm that renders a desired mode unobservable from a given set of sensor nodes. We derive optimality conditions and a heuristic optimization algorithm, which we validate through an example.**

I. INTRODUCTION

Network systems are broadly used to model engineering, social, and natural systems. An important property of such systems is their robustness to different contingencies, including failure of components affecting the flow of information, external disturbances altering individual node dynamics, and variations in the network topology and weights. It remains an outstanding problem to quantify how different topological features enable robustness, and to engineer complex network systems that ensure functionality and operability in the face of arbitrary and perhaps malicious perturbations.

Observability of a network system guarantees the ability to reconstruct the state of individual nodes from sparse measurements. While observability is a binary notion [1], the degree of observability (or controllability) of a network can be quantified in different ways, including the energy associated with the measurements [2], [3], the novelty of the output signal [4], the number of necessary sensor nodes [5], [6], [7], and the robustness to removal of interconnection edges [8]. A graded notion of observability is preferable, as it allows us to compare different networks, select optimal sensor nodes, and identify features favoring observability.

In this work we measure the robustness of a network based on the size of the smallest perturbation needed to prevent observability. Our notion of robustness is motivated by the fact that network weights are rarely known without uncertainty, and observability is a generic property [9]. For these reasons, numerical tests to assess observability may be unreliable and in fact fail to recognize unobservable systems: instead, our notion of observability, that is, the size of the smallest perturbation preventing observability or, equivalently, the distance to the nearest unobservable realization, can be measured more reliably [10]. Among our contributions, we highlight connections between the robustness of a network and its structure, and we propose an algorithmic procedure to construct optimal perturbations. Our work finds applicability, for instance, in network control problems where the network weights are time varying, in security applications where an attacker gains control of some network edges, and in network science for the classification of network edges and the design of robust complex networks.

Related work Our notion of robustness is inspired by classic works on the observability radius of dynamical systems [11], [12], [13], which is defined as the norm of the smallest perturbation yielding unobservability. In particular, for a linear system with matrices (A, C), the radius of observability is

$$\mu(A, C) = \min_{\Delta_A, \Delta_C} \left\| \begin{bmatrix} \Delta_A \\ \Delta_C \end{bmatrix} \right\|_2 ,$$
$$\text{s.t. } (A + \Delta_A, C + \Delta_C) \text{ is unobservable.}$$

As a known result [12], the observability radius satisfies

$$\mu(A, C) = \min_s \sigma_n \left(\begin{bmatrix} sI - A \\ C \end{bmatrix} \right),$$

where $s \in \mathbb{R}$ or $s \in \mathbb{C}$ if complex perturbations are allowed. The optimal perturbations Δ_A and Δ_C are typically full matrices and, to the best of our knowledge, all existing results and procedures are not applicable to the case where the perturbations must satisfy a desired sparsity constraint (e.g., see [14]). This scenario is in fact the relevant one for network systems, where the nonzero entries of the network matrices A and C correspond to existing network edges, and it would be undesirable or unrealistic for a perturbation to modify the interaction of disconnected nodes. A notable exception is [8], where, however, the discussion is limited to edge removal.

We depart from the literature by requiring the perturbation to be real, with a desired sparsity pattern, and confined to the network matrix, that is, $\Delta_C = 0$. Our approach builds on the theory of *total least squares* [15], [16]. With respect to existing results, our work proposes tailored procedures for network systems, fundamental bounds, and insights into the robustness of different network topologies. Although our presentation focuses on perturbations preventing observability, the extension to controllability is straightforward.

Contributions The contribution of this paper is twofold. First, we define a metric of network robustness that captures the resilience of a network system to structural per-

This material is based upon work supported in part by NSF awards #BCS-1430279 and #ECCS-1405330. Gianluca Bianchin and Fabio Pasqualetti are with the Mechanical Engineering Department, University of California at Riverside, gianluca@engr.ucr.edu, fabiopas@engr.ucr.edu. Andrea Gasparri is with the Department of Engineering, Roma Tre University, gasparri@dia.uniroma3.it. Paolo Frasca is with the Department of Applied Mathematics, University of Twente, p.frasca@utwente.nl.

978-1-4673-8682-1/$31.00 ©2016 AACC

turbations (Section II). Our metric evaluates the distance of a network from the set of unobservable networks with the same interconnection structure, and it extends existing works on the controllability and observability radius of linear systems. Second, we formulate an optimization problem to determine optimal perturbations (with smallest Frobenius norm) preventing observability. We show that the problem is not convex, derive optimality conditions, and show that any optimal solution solves a (nonlinear) generalized eigenvalue problem (Section III-A). Based on this analysis, we propose a numerical procedure based on the power iteration method to determine (sub)optimal solutions (Section III-B). Finally, we analytically compute optimal perturbations for three dimensional line networks (Section III-C), and we validate the effectiveness of our numerical procedure to work in practice.

II. Problem Setup and Preliminary Results

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively. Let $A = [a_{ij}]$ be the *weighted adjacency matrix* of $\mathcal{G}$, where $a_{ij} \in \mathbb{R}$ denotes the weight associated with the edge $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ whenever $(i, j) \notin \mathcal{E}$. Let e_i denote the i-th canonical vector of dimension n. Let $\mathcal{O} = \{o_1, \dots, o_p\} \subseteq \mathcal{V}$ be the set of *sensor nodes*, and define the network output matrix as

$$C_{\mathcal{O}} = \begin{bmatrix} e_{o_1} & \cdots & e_{o_p} \end{bmatrix}^{\mathsf{T}}.$$

Let $x_i(t) \in \mathbb{R}$ denote the *state* of node i at time t, and let $x : \mathbb{N}_{\geq 0} \to \mathbb{R}^n$ be the map describing the evolution over time of the network state. The network dynamics are described by the discrete time-invariant linear system model:

$$x(t+1) = Ax(t), \;\; y(t) = C_{\mathcal{O}} x(t), \tag{1}$$

where $y : \mathbb{N}_{\geq 0} \to \mathbb{R}^p$ is the output of the sensor nodes $\mathcal{O}$.

In this work we characterize structured network perturbations that prevent observability from the sensor nodes. To this aim, let $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ be the *constraint graph*, and define the set of matrices compatible with $\mathcal{H}$ as

$$\mathcal{A}_{\mathcal{H}} = \{M \;:\; M \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}, M_{ij} = 0 \text{ if } (i, j) \notin \mathcal{E}_{\mathcal{H}}\}.$$

Recall from the eigenvector observability test that the network (1) is observable if and only if there is no right eigenvector of A that lies in the kernel of $C_{\mathcal{O}}$, that is, $C_{\mathcal{O}} x \neq 0$ whenever $x \neq 0$, $Ax = \lambda x$, and $\lambda \in \mathbb{C}$ [21]. We consider and study the following optimization problem:

$$\begin{aligned} \min \quad & \|\Delta\|_{\mathrm{F}}^2, & \\ \text{s.t.} \quad & (A + \Delta)x = \lambda x, & \text{(eigenvalue constraint)}, \\ & \|x\|_2 = 1, & \text{(eigenvector constraint)}, \\ & C_{\mathcal{O}} x = 0, & \text{(unobservability)}, \\ & \Delta \in \mathcal{A}_{\mathcal{H}}, & \text{(structural constraint)}, \end{aligned} \tag{2}$$

where the minimization is carried out over the network perturbation $\Delta \in \mathbb{R}^{n \times n}$, the eigenvector $x \in \mathbb{C}^n$, and the unobservable eigenvalue $\lambda \in \mathbb{C}$. The cost function $\|\Delta\|_{\mathrm{F}}^2$ quantifies the total cost in the form of magnitude of edge perturbation to achieve unobservability, and $\mathcal{A}_{\mathcal{H}}$ encodes the desired sparsity pattern of the perturbation. It should be observed that (i) the minimization problem (2) is not convex because the variables Δ and x are multiplied by each other in the eigenvector constraint $(A + \Delta)x = \lambda x$, (ii) if $A \in \mathcal{A}_{\mathcal{H}}$, then the minimization problem is feasible if and only if there exists a network matrix $A + \Delta = \tilde{A} \in \mathcal{A}_{\mathcal{H}}$ satisfying the eigenvalue constraint, and (iii) if $\mathcal{H} = \mathcal{G}$, then the perturbation modifies the weights of existing edges only.

We make the following assumption, which implies that Δ must be nonzero to satisfy the constraints in (2):

(A1) The pair $(A, C_{\mathcal{O}})$ is observable.

The minimization problem (2) can be solved by two subsequent steps. First, we fix the eigenvalue λ, and compute an optimal perturbation that solves the minimization problem for the fixed eigenvalue. This computation is the topic of the next section. Second, we search the complex plane for the optimal eigenvalue λ yielding the perturbation with minimum cost. We observe that (i) the exhaustive search of the optimal eigenvalue λ is an inherent feature of this class of problems, as also highlighted in prior work [13]; (ii) in some cases and for certain network topologies the optimal λ can be found analytically; and (iii) in several practical applications, the choice of λ is guided by the structure of the problem or it is a given constraint in the optimization.

III. Optimality Conditions and Algorithms for the Network Observability Radius

We now consider problem (2) with a fixed choice for λ. Let Δ^* be an optimal solution to (2). Then, we refer to $\|\Delta^*\|_{\mathrm{F}}^2$ as to the observability radius of the network A with sensor nodes $\mathcal{O}$, constraint graph $\mathcal{H}$, and unobservable eigenvalue λ.

A. Optimal network perturbation

In this section we manipulate the minimization problem (2) to facilitate its solution. Without affecting generality, relabel the network nodes such that the sensor nodes satisfy

$$\mathcal{O} = \{1, \dots, p\}, \text{ so that } C_{\mathcal{O}} = \begin{bmatrix} I_p & 0 \end{bmatrix}. \tag{3}$$

Accordingly,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ and } \Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}, \tag{4}$$

where $A_{11} \in \mathbb{R}^{p \times p}$, $A_{12} \in \mathbb{R}^{p \times n-p}$, $A_{21} \in \mathbb{R}^{n-p \times p}$, and $A_{22} \in \mathbb{R}^{n-p \times n-p}$. Let $V = [v_{ij}]$ be the unweighted adjacency matrix of $\mathcal{H}$, where $v_{ij} = 1$ if $(i, j) \in \mathcal{E}_{\mathcal{H}}$, and $v_{ij} = 0$ otherwise. Following the partitioning of A in (4), let

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

We perform the following three simplifying steps.

(1 – Rewriting the structural constraints) Let $B = A + \Delta$, and notice that $\|\Delta\|_{\mathrm{F}}^2 = \sum_{i=1}^n \sum_{j=1}^n (b_{ij} - a_{ij})^2$. Then, the minimization problem (2) can equivalently be rewritten restating the constraint $\Delta \in \mathcal{A}_{\mathcal{H}}$, as in the following equality:

$$\|\Delta\|_{\mathrm{F}}^2 = \|B - A\|_{\mathrm{F}}^2 = \sum_{i=1}^n \sum_{j=1}^n (b_{ij} - a_{ij})^2 v_{ij}^{-1}.$$

Notice that $\|\Delta\|_F^2 = \infty$ whenever Δ does not satisfy the structural constraint, that is, when $v_{ij} = 0$ and $b_{ij} \neq a_{ij}$.

(2 – Minimization with real variables) Let $\lambda = \lambda_\Re + \mathrm{i}\lambda_\Im$, where i denotes the imaginary unit. Let

$$x_\Re = \begin{bmatrix} x_\Re^1 \\ x_\Re^2 \end{bmatrix}, \text{ and } x_\Im = \begin{bmatrix} x_\Im^1 \\ x_\Im^2 \end{bmatrix},$$

denote the real and imaginary parts of the eigenvector x, with $x_\Re^1 \in \mathbb{R}^p$, $x_\Im^1 \in \mathbb{R}^p$, $x_\Re^2 \in \mathbb{R}^{n-p}$, and $x_\Im^2 \in \mathbb{R}^{n-p}$. We now prove that the minimization problem (2) can be restated and solved over real variables only.

Lemma 3.1: ***(Minimization with real eigenvector constraint)*** The constraint $(A + \Delta)x = \lambda x$ can equivalently be written as

$$\begin{aligned} (A + \Delta - \lambda_\Re I)x_\Re &= -\lambda_\Im x_\Im, \\ (A + \Delta - \lambda_\Re I)x_\Im &= \lambda_\Im x_\Re. \end{aligned} \tag{5}$$

Proof: By considering separately the real and imaginary part of the eigenvalue constraint, we have $(A+\Delta)x = \lambda_\Re x + \mathrm{i}\lambda_\Im x$ and $(A + \Delta)\bar{x} = \lambda_\Re x - \mathrm{i}\lambda_\Im \bar{x}$, where $\bar{x}$ denotes the complex conjugate of x. Notice that

$$\underbrace{(A + \Delta)(x + \bar{x})}_{(A+\Delta)2x_\Re} = \underbrace{(\lambda_\Re + \mathrm{i}\lambda_\Im)x + (\lambda_\Re - \mathrm{i}\lambda_\Im)\bar{x}}_{2\lambda_\Re x_\Re - 2\lambda_\Im x_\Im},$$

and, analogously,

$$\underbrace{(A + \Delta)(x - \bar{x})}_{(A+\Delta)2\mathrm{i}x_\Im} = \underbrace{(\lambda_\Re + \mathrm{i}\lambda_\Im)x - (\lambda_\Re - \mathrm{i}\lambda_\Im)\bar{x}}_{2\mathrm{i}\lambda_\Re x_\Im + 2\mathrm{i}\lambda_\Im x_\Re},$$

which concludes the proof. ■

(3 – Reduction of dimensionality) The constraint $C_\mathcal{O} x = 0$ and equation (3) imply that $x_\Re^1 = x_\Im^1 = 0$. Thus, in the minimization problem (4) we set $\Delta_{11} = 0$, $\Delta_{21} = 0$, and consider only the minimization variables $x_\Re^2$, $x_\Im^2$, Δ_{12}, and Δ_{22}. These simplifications lead to the following result.

Lemma 3.2: ***(Equivalent minimization problem)*** Let

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, \bar{\Delta} = \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \end{bmatrix}, \bar{M} = \begin{bmatrix} 0_{p\times n-p} \\ \lambda_\Im I_{n-p} \end{bmatrix}, \\ \bar{N} &= \begin{bmatrix} 0_{p\times n-p} \\ \lambda_\Re I_{n-p} \end{bmatrix}, \bar{V} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}, \text{ and } \bar{B} = \bar{A} + \bar{\Delta}. \end{aligned} \tag{6}$$

The following minimization problem is equivalent to (2):

$$\begin{aligned} \|\bar{\Delta}^*\|_F^2 = \min_{\bar{B}, x_\Re^2, x_\Im^2} \quad & \textstyle\sum_{i=1}^n \sum_{p+1}^n (\bar{b}_{ij} - \bar{a}_{ij})^2 v_{ij}^{-1}, \\ \text{s.t.} \quad & \begin{bmatrix} \bar{B} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{B} - \bar{N} \end{bmatrix} \begin{bmatrix} x_\Re^2 \\ x_\Im^2 \end{bmatrix} = 0, \\ & \left\| \begin{bmatrix} x_\Re^2 \\ x_\Im^2 \end{bmatrix} \right\|_2 = 1. \end{aligned} \tag{7}$$

The minimization problem (7) belongs to the class of *(structured) total least squares* problems, which arise in several estimation and identification problems in control theory and signal processing. Our approach is inspired by [16], with the difference that we focus on real perturbations Δ and complex eigenvalue λ: this choice leads to different optimality conditions and algorithms. Let $A \otimes B$ denote the Kronecker product between A and B, and $\mathrm{diag}(d_1, \dots, d_n)$ the diagonal matrix with entries $d_1, \dots, d_n$. We now derive the optimality conditions for the minimization problem (7).

Theorem 3.3: ***(Optimality conditions)*** Let $x_\Re^*$, and $x_\Im^*$ be a solution to the minimization problem (7). Then,

$$\begin{aligned} \underbrace{\begin{bmatrix} \bar{A} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{A} - \bar{N} \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_\Re^* \\ x_\Im^* \end{bmatrix}}_{x^*} &= \sigma \underbrace{\begin{bmatrix} S_x & T_x \\ T_x & Q_x \end{bmatrix}}_{D_x} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y^*}, \\ \underbrace{\begin{bmatrix} \bar{A} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{A} - \bar{N} \end{bmatrix}^\mathsf{T}}_{\tilde{A}^\mathsf{T}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y^*} &= \sigma \underbrace{\begin{bmatrix} S_y & T_y \\ T_y & Q_y \end{bmatrix}}_{D_y} \underbrace{\begin{bmatrix} x_\Re^* \\ x_\Im^* \end{bmatrix}}_{x^*}, \end{aligned} \tag{8}$$

for some $\sigma > 0$ and $y^* \in \mathbb{R}^{2n}$ with $\|y^*\| = 1$, and where

$$\begin{aligned} &D_1 = \mathrm{diag}(v_{11}, \dots, v_{1n}, v_{21}, \dots, v_{2n}, \dots, v_{n1}, \dots, v_{nn}), \\ &D_2 = \mathrm{diag}(v_{11}, \dots, v_{n1}, v_{12}, \dots, v_{n2}, \dots, v_{1n}, \dots, v_{nn}), \\ &S_x = (I \otimes x_\Re^*)^\mathsf{T} D_1 (I \otimes x_\Re^*), T_x = (I \otimes x_\Re^*)^\mathsf{T} D_1 (I \otimes x_\Im^*), \\ &Q_x = (I \otimes x_\Im^*)^\mathsf{T} D_1 (I \otimes x_\Im^*), S_y = (I \otimes y_1)^\mathsf{T} D_2 (I \otimes y_1), \\ &T_y = (I \otimes y_1)^\mathsf{T} D_2 (I \otimes y_2), Q_y = (I \otimes y_2)^\mathsf{T} D_2 (I \otimes y_2). \end{aligned} \tag{9}$$

Proof: We adopt the method of Lagrange multipliers to derive the optimality conditions for the minimization problem (7). The Lagrangian is

$$\begin{aligned} \mathcal{L}(\bar{B}, x_\Re^2, x_\Im^2, \ell_1, \ell_2, \rho) &= \sum_i \sum_j (\bar{b}_{ij} - \bar{a}_{ij})^2 v_{ij}^{-1} \\ &+ \ell_1^\mathsf{T}((\bar{B} - \bar{N})x_\Re^2 + \bar{M}x_\Im^2) + \ell_2^\mathsf{T}((\bar{B} - \bar{N})x_\Im^2 - \bar{M}x_\Re^2) \\ &+ \rho(1 - x_\Re^{2\mathsf{T}} x_\Re^2 - x_\Im^{2\mathsf{T}} x_\Im^2), \end{aligned} \tag{10}$$

where $\ell_1 \in \mathbb{R}^n$, $\ell_2 \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$ are Lagrange multipliers. By equating the partial derivatives of $\mathcal{L}$ to zero we obtain

$$\frac{\partial \mathcal{L}}{\partial b_{ij}} = 0 \Rightarrow -2(\bar{a}_{ij} - \bar{b}_{ij})v_{ij}^{-1} + \ell_{1i} x_{\Re j}^2 + \ell_{2i} x_{\Im j}^2 = 0, \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial x_\Re^2} = 0 \Rightarrow \ell_1^\mathsf{T}(\bar{B} - \bar{N}) - \ell_2^\mathsf{T}\bar{M} - 2\rho x_\Re^{2\mathsf{T}} = 0, \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial x_\Im^2} = 0 \Rightarrow \ell_1^\mathsf{T}\bar{M} + \ell_2^\mathsf{T}(\bar{B} - \bar{N}) - 2\rho x_\Im^{2\mathsf{T}} = 0, \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_1} = 0 \Rightarrow (\bar{B} - \bar{N})x_\Re^2 + \bar{M}x_\Im^2 = 0, \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_2} = 0 \Rightarrow (\bar{B} - \bar{N})x_\Im^2 - \bar{M}x_\Re^2 = 0, \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = 0 \Rightarrow x_\Re^{2\mathsf{T}} x_\Re^2 + x_\Im^{2\mathsf{T}} x_\Im^2 = 1. \tag{16}$$

Let $L_1 = \mathrm{diag}(\ell_1)$, $L_2 = \mathrm{diag}(\ell_2)$, $X_\Re = \mathrm{diag}(x_\Re^2)$, $X_\Im = \mathrm{diag}(x_\Im^2)$. After including the factor 2 into the multipliers, equation (11) can be written in matrix form as

$$\bar{A} - \bar{B} = -\Delta = L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im. \tag{17}$$

Analogously, equations (12) and (13) can be written as

$$\begin{bmatrix} \ell_1^\mathsf{T} & \ell_2^\mathsf{T} \end{bmatrix} \begin{bmatrix} \bar{B} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{B} - \bar{N} \end{bmatrix} - 2\rho \begin{bmatrix} x_\Re^{2\mathsf{T}} & x_\Im^{2\mathsf{T}} \end{bmatrix} = 0. \tag{18}$$

From equation (18) we have

$$\begin{bmatrix}\ell_1^\mathsf{T} & \ell_2^\mathsf{T}\end{bmatrix} \underbrace{\begin{bmatrix}\bar{B}-\bar{N} & \bar{M} \\ -\bar{M} & \bar{B}-\bar{N}\end{bmatrix}\begin{bmatrix}x_\Re^2 \\ x_\Im^2\end{bmatrix}}_{=0 \text{ due to (14) and (15)}} - 2\rho = 0,$$

from which we conclude that $\rho = 0$. By combining (14) and (17) (respectively, (15) and (17)) we obtain

$$\begin{aligned}(\bar{A}-\bar{N})x_\Re^2 + \bar{M}x_\Im^2 &= \left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right)x_\Re^2, \\ (\bar{A}-\bar{N})x_\Im^2 - \bar{M}x_\Re^2 &= \left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right)x_\Im^2.\end{aligned}$$

Analogously, by combining (12) and (17) (respectively, (13) and (17)) we obtain

$$\begin{aligned}\ell_1^\mathsf{T}(\bar{A}-\bar{N}) - \ell_2^\mathsf{T}\bar{M} &= \ell_1^\mathsf{T}\left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right), \\ \ell_2^\mathsf{T}(\bar{A}-\bar{N}) + \ell_1^\mathsf{T}\bar{M} &= \ell_2^\mathsf{T}\left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right).\end{aligned}$$

Let $\sigma = \sqrt{\ell_1^\mathsf{T}\ell_1 + \ell_2^\mathsf{T}\ell_2}$, and observe that σ cannot be zero. Indeed, due to Assumption (A1), the optimal perturbation can not be zero; thus, the first constraint in (7) must be active and the corresponding multiplier must be nonzero. Then, we define $y_1 = \ell_1/\sigma$ and $y_2 = \ell_2/\sigma$, and verify that

$$\begin{aligned}\left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right)x_\Re^2 &= \sigma\left(S_x y_1 + T_x y_2\right), \\ \left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right)x_\Im^2 &= \sigma\left(T_x y_1 + Q_x y_2\right),\end{aligned}$$

and

$$\begin{aligned}\sigma\left(y_1^\mathsf{T}(\bar{A}-\bar{N}) - y_2^\mathsf{T}\bar{M}\right) &= \ell_1^\mathsf{T}\left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right) \\ &= \sigma^2\left(S_y x_\Re^2 + T_y x_\Im^2\right)^\mathsf{T}, \\ \sigma\left(y_2^\mathsf{T}(\bar{A}-\bar{N}) + y_1^\mathsf{T}\bar{M}\right) &= \ell_2^\mathsf{T}\left(L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\right) \\ &= \sigma^2\left(T_y x_\Re^2 + Q_y x_\Im^2\right)^\mathsf{T},\end{aligned}$$

which conclude the proof. ■

Note that equations (8) may admit multiple solutions, and that every solution to (8) yields a network perturbation that satisfies the constraints in the minimization problem (7). We now state a result to compute of feasible perturbations.

Corollary 3.4: ***(Minimum norm perturbation)*** Let Δ^* be a solution to (2). Then, $\Delta^* = [0^{n\times p}\ \bar{\Delta}^*]$, where

$$\bar{\Delta}^* = -\sigma\left(\mathrm{diag}(y_1)\bar{V}\mathrm{diag}(x_\Re^*) - \mathrm{diag}(y_2)\bar{V}\mathrm{diag}(x_\Im^*)\right),$$

and $x_\Re^*$, $x_\Im^*$, y_1, y_2, σ satisfy the equations (8). Moreover,

$$\|\Delta\|_\mathrm{F}^2 = \sigma^2 x^{*\mathsf{T}} D_y x^* = \sigma x^{*\mathsf{T}}\tilde{A}^\mathsf{T} y^* \leq \sigma\|\tilde{A}\|_\mathrm{F}.$$

Proof: The expression for the perturbation Δ^* comes from Lemma 3.2 and (17), and the fact that $L_1 = \sigma\,\mathrm{diag}(y_1)$, $L_2 = \sigma\,\mathrm{diag}(y_2)$. To show the second part, notice that

$$\begin{aligned}\|\Delta\|_\mathrm{F}^2 = \|A-B\|_\mathrm{F}^2 &= \|L_1\bar{V}X_\Re + L_2\bar{V}X_\Im\|_\mathrm{F}^2 \\ &= \sigma^2\sum_i\sum_j\left(y_{1i}^2 x_{\Re j}^2 + y_{2i}^2 x_{\Im j}^2\right)v_{ij} \\ &= \sigma^2 x^{*\mathsf{T}} D_y x^* = \sigma x^{*\mathsf{T}}\tilde{A}^\mathsf{T} y^*,\end{aligned}$$

where the last equalities follow from (8). Finally, the theorem follows from $\|x^*\|_2 = \|x^*\|_\mathrm{F} = \|y^*\|_2 = \|y^*\|_\mathrm{F} = 1$. ■

To compute a triple (σ, x^*, y^*) satisfying the condition in Theorem 3.3, observe that equations (8) can be written in matrix form as

$$\underbrace{\begin{bmatrix}0 & \tilde{A}^\mathsf{T} \\ \tilde{A} & 0\end{bmatrix}}_{H}\underbrace{\begin{bmatrix}x \\ y\end{bmatrix}}_{z} = \bar{\sigma}\underbrace{\begin{bmatrix}D_y & 0 \\ 0 & D_x\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}x \\ y\end{bmatrix}}_{z}. \qquad (19)$$

Lemma 3.5: ***(Equivalence between Theorem 3.3 and (19))*** Let (σ, x, y), with $x \neq 0$, solve (19). Then, $\sigma \neq 0$ and $y \neq 0$, and the triple $((\alpha\beta)^{-1}\sigma, \alpha x, \beta y)$, with $\alpha = \mathrm{sgn}(\sigma)\|x\|^{-1}$ and $\beta = \|y\|^{-1}$, satisfies the conditions in Theorem 3.3.

Proof: Because $x \neq 0$ and $\tilde{A}$ has full column rank due to Assumption (A1), it follows that $\sigma \neq 0$ and $y \neq 0$. Let D_x and D_y be as in (8). Notice that $D_{\alpha x} = \alpha^2 D_x$ and $D_{\beta y} = \beta^2 D_y$. Notice that $(\alpha\beta)^{-1}\sigma > 0$. We obtain that

$$\begin{aligned}\tilde{A}\alpha x &= \frac{\sigma}{\alpha\beta}\alpha^2 D_x \beta y = \alpha\sigma D_x y, \\ \tilde{A}^\mathsf{T}\beta y &= \frac{\sigma}{\alpha\beta}\beta^2 D_y \alpha x = \beta\sigma D_y x,\end{aligned}$$

which concludes the proof. ■

Lemma 3.5 shows that a (sub)optimal network perturbation can in fact be constructed by solving equations (19). It should be observed that, if the matrices S_x, T_x, Q_x, S_y, T_y, and Q_y were constant, then (19) would describe a generalized eigenvalue problem, and a solution $(\bar{\sigma}, z)$ would be a pair of generalized eigenvalue and eigenvector. These facts will be exploited in the next section to develop a heuristic algorithm to compute a (sub)optimal network perturbation.

B. A heuristic procedure to compute structural perturbations

In this section we propose an algorithm to find a solution to the set of nonlinear equations (19), and therefore to find a (sub)optimal solution to the minimization problem (2). Our procedure is motivated by (19) and Corollary 3.4: at each iteration, we fix a vector z, compute the corresponding matrix D, and approximate an eigenvector associated with the smallest generalized eigenvalue of the pair (H, D). Because the size of the perturbation is bounded by the generalized eigenvalue σ as motivated in Corollary 3.4, we adopt an iterative procedure based on the *inverse iteration* method for the computation of the smallest eigenvalue of a matrix [22]. We remark that our procedure is heuristic, because (19) is in fact a nonlinear generalized eigenvalue problem due to the dependency of the matrix D on the eigenvector z.

We start by characterizing certain properties of H and D, which will be used to derive our algorithm. Let

$$\mathrm{spec}(H, D) = \{\lambda \in \mathbb{C} \ :\ \det(H - \lambda D) = 0\},$$

and recall that the pencil (H, D) is regular if the determinant $\det(H - \lambda D)$ does not vanish for all values of λ [23]. Notice that, if (H, D) is not regular, then $\mathrm{spec}(H, D) = \mathbb{C}$.

Lemma 3.6: ***(Generalized eigenvalues of (H, D))*** Given a vector $z \in \mathbb{R}^{2n+2p}$, define the matrices H and D as in (19). Then,

(i) $0 \in \mathrm{spec}(H, D)$;

(ii) if $\lambda \in \text{spec}(H, D)$, then $-\lambda \in \text{spec}(H, D)$; and
(iii) if (H, D) is regular, then $\text{spec}(H, D) \subset \mathbb{R}$.

Proof: Notice that statement (i) is equivalent to $\tilde{A}x = 0$ and $\tilde{A}^\mathsf{T} y = 0$, for some vectors x and y. Because $\tilde{A}^\mathsf{T} \in \mathbb{R}^{(2n-2p)\times 2n}$ with $p \geq 1$, the matrix $\tilde{A}^\mathsf{T}$ features a nontrivial null space. Thus, the two equations are satisfied with $x = 0$ and $y \in \text{Ker}(\tilde{A}^\mathsf{T})$, and the claimed statement follows.

To prove statement (ii) notice that, due to the block structure of H and D, if the triple $(\lambda, \bar{x}, \bar{y})$ satisfies the generalized eigenvalue equations $\tilde{A}^\mathsf{T}\bar{y} = \lambda D_y \bar{x}$ and $\tilde{A}\bar{x} = \lambda D_x \bar{y}$, so does the triple $(-\lambda, \bar{x}, -\bar{y})$.

To show statement (iii), let $\text{Rank}(D) = k \leq n$, and notice that the regularity of the pencil (H, D) implies $H\bar{z} \neq 0$ whenever $D\bar{z} = 0$ and $\bar{z} \neq 0$. Notice that (H, D) has $n - k$ infinite eigenvalues [23] because $H\bar{z} = \lambda D\bar{z} = \lambda \cdot 0$ for every nontrivial $\bar{z} \in \text{Ker}(D)$. Because D is symmetric, it admits an orthonormal basis of eigenvectors. Let $V_1 \in \mathbb{R}^{n\times k}$ contain the orthonormal eigenvectors of D associated with its nonzero eigenvalues, let Λ_D be the corresponding diagonal matrix of the eigenvalues, and let $T_1 = V_1 \Lambda_D^{-1/2}$. Then, $T_1^\mathsf{T} D T_1 = I$. Let $\tilde{H} = T_1^\mathsf{T} H T_1$, and notice that $\tilde{H}$ is symmetric. Let $T_2 \in \mathbb{R}^{k\times k}$ be an orthonormal matrix of the eigenvectors of $\tilde{H}$. Let $T = T_1 T_2$ and note that

$$T^\mathsf{T} H T = \Lambda, \text{ and } T^\mathsf{T} D T = I,$$

where Λ is a diagonal matrix. To conclude, consider the generalized eigenvalue problem $H\bar{z} = \lambda D\bar{z}$. Let $\bar{z} = T\tilde{z}$. Because T has full column rank k, we have

$$T^\mathsf{T} H T\tilde{z} = \Lambda\tilde{z} = \lambda T^\mathsf{T} D T\tilde{z} = \lambda\tilde{z},$$

which implies that (H, D) has k real eigenvalues. ■

Lemma 3.6 implies that the inverse iteration method is not directly applicable to (19). In fact, the zero eigenvalue of (H, D) leads the inverse iteration to instability, while the presence of eigenvalues of (H, D) with equal magnitude may induce non-decaying oscillations in the solution vector. To overcome these issues, we employ a shifting mechanism as detailed in Algorithm 1, where the eigenvector z is iteratively updated by solving the equation $(H - \mu D)z_{k+1} = Dz_k$ until a convergence criteria is met. Notice that (i) the eigenvalues of $(H - \mu D, D)$ are shifted with respect to the eigenvalues of (H, D), that is, if $\sigma \in \text{spec}(H, D)$, then $\sigma + \mu \in \text{spec}(H - \mu D, D)$, (ii) the pairs $(H - \mu D, D)$ and (H, D) share the same eigenvectors, and (iii) by selecting $\mu = \psi \cdot \min\{\sigma \in \text{spec}(H, D) \ : \ \sigma > 0\}$, the pair $(H - \mu D, D)$ has nonzero eigenvalues with distinct magnitude. Thus, Algorithm 1 estimates the eigenvector z associated with the smallest nonzero eigenvalue σ of (H, D), and converges when z and σ also satisfy equations (19). The parameter ψ determines a compromise between numerical stability and convergence speed; larger values of ψ improve the convergence speed.

When convergent, Algorithm 1 finds a solution to (19) and, consequently, it allows to compute a (sub)optimal network perturbation preventing observability of a desired eigenvalue. All information about the network matrix, the sensor nodes, the constraint graph, and the unobservable eigenvalue is encoded in the matrix H according to the definitions (6), (8) and (19). Although convergence of Algorithm 1 is not guaranteed, numerical studies show that it performs well in practice; see Fig. 1 for a numerical validation.

Algorithm 1: Heuristic solution to (19)

Input: Matrix H; max iterations $\max_{\text{iter}}$; $\psi \in (0.5, 1)$.
Output: σ and z satisfying (19), or fail.
repeat
  $z \leftarrow (H - \mu D)^{-1} Dz$;
  $\sigma \leftarrow \|z\|$;
  $z \leftarrow z/\sigma$;
  $\mu = \psi \cdot \min\{\sigma \in \text{spec}(H, D) \ : \ \sigma > 0\}$;
  update D according to (9);
  $i \leftarrow i + 1$
until *convergence or* $i > \max_{\text{iter}}$;
return $(\sigma + \mu, z)$ *or fail if* $i = \max_{\text{iter}}$;

C. Optimal perturbations and algorithm validation

In this section validate Algorithm 1 on a three nodes line network. We first provide the following analytical result on optimal perturbations of three dimensional line networks.

Theorem 3.7: ***(Optimal perturbations of 3-dimensional line networks with fixed*** $\lambda \in \mathbb{C}$***)*** Consider a network with graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $|\mathcal{V}| = 3$, weighted adjacency matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

and sensor node $\mathcal{O} = \{1\}$. Let $B = [b_{ij}] = A + \Delta^*$, where Δ^* solves the minimization problem (2) with $\mathcal{H} = \mathcal{G}$ and $\lambda = \lambda_\Re + \mathrm{i}\lambda_\Im \in \mathbb{C}$ with $\lambda_\Im \neq 0$. Then:

$$b_{11} = a_{11}, \quad b_{21} = a_{21}, \quad b_{12} = 0,$$

and b_{22}, b_{23}, b_{32}, b_{33}, satisfy:

$$\begin{aligned} (b_{22} - a_{22}) - (b_{33} - a_{33}) + \frac{b_{33} - b_{22}}{b_{32}}(b_{23} - a_{23}) &= 0, \\ (b_{32} - a_{32}) - \frac{b_{23}}{b_{32}}(b_{23} - a_{23}) &= 0, \\ 2\lambda_\Re + b_{22} + b_{33} &= 0, \\ b_{22}b_{33} - b_{23}b_{32} - \lambda_\Re^2 + \lambda_\Im^2 &= 0. \end{aligned} \tag{20}$$

Proof: Let x satisfy $Bx = \lambda x$ and notice that, because λ is unobservable, $C_\mathcal{O} x = [1 \ 0 \ 0]x = 0$. Then, $x = [x_1 \ x_2 \ x_3]^\mathsf{T}$, $x_1 = 0$, $b_{11} = a_{11}$, and $b_{21} = a_{21}$.

We now show that $x_2 \neq 0$. By contradiction and let $x_2 = 0$. Notice that the relation $Bx = \lambda x$ implies $b_{33} = \lambda$, which contradicts the assumption that $\lambda_\Im \neq 0$ and $b_{33} \in \mathbb{R}$.

Because $x_2 \neq 0$, the relation $Bx = \lambda x$ and $x_1 = 0$ imply $b_{12} = 0$. Additionally, λ is an eigenvalue of

$$B_2 = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}.$$

The characteristic polynomial of B_2 is

$$P_{B_2}(s) = s^2 - (b_{22} + b_{33})s + b_{22}b_{33} - b_{23}b_{32}.$$

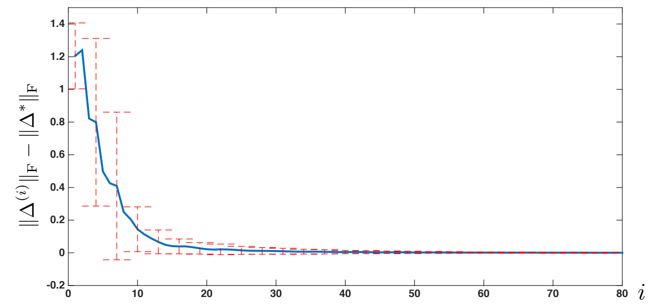

Fig. 1. This figure validates effectiveness of Algorithm 1 in computing optimal perturbations for the line network in Section III-C. The plot shows mean and standard deviation over 100 networks of the difference between Δ^*, obtained via optimality conditions (20), and $\Delta^{(i)}$, computed at the i-th iteration of Algorithm 1. The unobservable eigenvalue is $\lambda = \mathrm{i}$ and the values a_{ij} are chosen independently and uniformly distributed in $[0, 1]$.

Then, for λ to be an eigenvalue of B_2, we must have

$$\begin{aligned} -b_{22} - b_{33} - 2\lambda_{\Re} = 0, \text{ and} \\ b_{22}b_{33} - b_{23}b_{32} - \lambda_{\Re}^2 + \lambda_{\Im}^2 = 0. \end{aligned} \tag{21}$$

The Lagrange function of the minimization problem with cost function $\|\Delta^*\|_F^2 = \sum_{i=2}^{3}\sum_{j=2}^{3}(b_{ij} - a_{ij})^2$ and constraints (21) reads as

$$\begin{aligned} \mathcal{L}(b_{22}, b_{23}, b_{32}, b_{33}, p_1, p_2) &= d_{22}^2 + d_{23}^2 + d_{32}^2 + d_{33}^2 \\ &+ p_1(2\lambda_{\Re} + b_{22} + b_{33}) \\ &+ p_2(b_{22}b_{33} - b_{23}b_{32} - (\lambda_{\Re}^2 + \lambda_{\Im}^2)), \end{aligned}$$

where $p_1 \in \mathbb{R}$ and $p_2 \in \mathbb{R}$ are Lagrange multipliers, and $d_{ij} = b_{ij} - a_{ij}$. By equating the partial derivatives of $\mathcal{L}$ to zero we obtain

$$\frac{\partial \mathcal{L}}{\partial b_{22}} = 0 \Rightarrow 2d_{22} + p_1 + p_2 b_{33} = 0, \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial b_{33}} = 0 \Rightarrow 2d_{33} + p_1 + p_2 b_{22} = 0, \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial b_{23}} = 0 \Rightarrow 2d_{23} - p_2 b_{32} = 0, \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial b_{32}} = 0 \Rightarrow 2d_{32} - p_2 b_{23} = 0, \tag{25}$$

together with (21). The statement follows by substitutions of the Lagrange multipliers p_1 and p_2 into (22) and (25). ■

To validate Algorithm 1, we compute optimal perturbations for 3-dimensional line networks based on Theorem 3.7, and we compare it with the perturbation obtained at different iterations of Algorithm 1. As shown in Fig. 1, Algorithm 1 determines a network perturbation with minimum norm.

IV. Conclusion

In this work we introduce the notion of observability radius for network systems, which measures the ability to maintain observability of the network modes against perturbations of the edge weights. The paper contains two sets of results. On the one hand, we perform a rigorous analysis to characterize network perturbations preventing observability and describe a heuristic algorithm to compute perturbations with minimum Frobenius norm. On the other hand, we explicitly characterize the observability radius of three dimensional line networks, and we validate our heuristic algorithm. Several aspects are left as the subject of future investigation, including a characterization of the observability radius of random networks, the computation of eigenvalues requiring perturbations with minimum norm, and a study of the relation between the topology of a network and its observability robustness to structured perturbations.

References

[1] R. E. Kalman, Y. C. Ho, and S. K. Narendra, "Controllability of linear dynamical systems," *Contributions to Differential Equations*, vol. 1, no. 2, pp. 189–213, 1963.

[2] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 40–52, 2014.

[3] F. L. Cortesi, T. H. Summers, and J. Lygeros, "Submodularity of energy related controllability metrics," in *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 2883–2888.

[4] G. Kumar, D. Menolascino, and S. Ching, "Input novelty as a control metric for time varying linear systems," *arXiv preprint arXiv:1411.5892*, 2014.

[5] A. Olshevsky, "Minimal controllability problems," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 3, pp. 249–258, Sept 2014.

[6] S. Pequito, G. Ramos, S. Kar, A. P. Aguiar, and J. Ramos, "On the exact solution of the minimal controllability problem," *arXiv preprint arXiv:1401.4209*, 2014.

[7] N. Monshizadeh, S. Zhang, and M. K. Camlibel, "Zero forcing sets and controllability of dynamical systems defined on graphs," *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2562–2567, 2014.

[8] M. Fardad, A. Diwadkar, and U. Vaidya, "On optimal link removals for controllability degradation in dynamical networks," in *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 499–504.

[9] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. Springer, 1985.

[10] C. Paige, "Properties of numerical algorithms related to computing controllability," *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 130–138, 1981.

[11] R. Eising, "Between controllable and uncontrollable," *Systems & Control Letters*, vol. 4, no. 5, pp. 263–264, 1984.

[12] D. L. Boley and W.-S. Lu, "Measuring how far a controllable system is from an uncontrollable one," *IEEE Transactions on Automatic Control*, vol. 31, no. 3, pp. 249–251, 1986.

[13] G. Hu and E. J. Davison, "Real controllability/stabilizability radius of lti systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 254–257, 2004.

[14] M. Karow and D. Kressner, "On the structured distance to uncontrollability," *Systems & Control Letters*, vol. 58, no. 2, pp. 128–132, 2009.

[15] Y. Nievergelt, "Total least squares: State-of-the-art regression in numerical analysis," *SIAM Review*, vol. 36, no. 2, pp. 258–264, 1994.

[16] B. De Moor, "Total least squares for affinely structured matrices and the noisy realization problem," *IEEE Transactions on Signal Processing*, vol. 42, no. 11, pp. 3104–3113, 1994.

[17] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 162–186, 2009.

[18] A. Chapman and M. Mesbahi, "On symmetry and controllability of multi-agent systems," in *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 625–630.

[19] F. Pasqualetti and S. Zampieri, "On the controllability of isotropic and anisotropic networks," in *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 607–612.

[20] A. J. Whalen, S. N. Brennan, T. D. Sauer, and S. J. Schiff, "Observability and controllability of nonlinear networks: The role of symmetry," *Physical Review X*, vol. 5, no. 1, p. 011005, 2015.

[21] T. Kailath, *Linear Systems*. Prentice-Hall, 1980.

[22] L. N. Trefethen and D. Bau III, *Numerical linear algebra*. Siam, 1997, vol. 50.

[23] G. H. Golub and C. F. Van Loan, *Matrix computations*. JHU Press, 2012.