Data-driven Learning of Minimum-Energy Controls for Linear Systems

Giacomo Baggio, Vaibhav Katewa, and Fabio Pasqualetti

***Abstract*— In this paper we study the problem of learning minimum-energy controls for linear systems from experimental data. The design of open-loop minimum-energy control inputs to steer a linear system between two different states in finite time is a classic problem in control theory, whose solution can be computed in closed form using the system matrices and its controllability Gramian. Yet, the computation of these inputs is known to be ill-conditioned, especially when the system is large, the control horizon long, and the system model uncertain. Due to these limitations, open-loop minimum-energy controls and the associated state trajectories have remained primarily of theoretical value. Surprisingly, in this paper we show that open-loop minimum-energy controls can be learned exactly from experimental data, with a finite number of control experiments over the same time horizon, without knowledge or estimation of the system model, and with an algorithm that is significantly more reliable than the direct model-based computation. These findings promote a new philosophy of controlling large, uncertain, linear systems where data is abundantly available.**

I. Introduction

Consider the discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where, respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ denote the system and input matrices, and $x : \mathbb{N} \to \mathbb{R}^n$ and $u : \mathbb{N} \to \mathbb{R}^m$ describe the state and input of the system. For a control horizon $T \in \mathbb{N}$ and a desired state x_f, the minimum-energy control problem asks for the input sequence $u(0), \dots, u(T-1)$ with minimum energy that steers the state from $x(0)$ to x_f in T steps, and it can be formulated as

$$\begin{aligned} \min_{u} \quad & \sum_{t=0}^{T-1} \|u(t)\|_2^2, \\ \text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t), \\ & x(T) = x_\mathrm{f}. \end{aligned} \tag{2}$$

As a classic result [1], the minimization problem (2) is feasible if and only if $(x_\mathrm{f} - A^T x(0)) \in \mathrm{Im}(W_T)$, where

$$W_T = \sum_{t=0}^{T-1} A^t B B^\mathsf{T} (A^\mathsf{T})^t \tag{3}$$

is the T-steps controllability Gramian and $\mathrm{Im}(W_T)$ denotes the image of the matrix W_T. Further, the solution to (2) is

$$u^*(t) = B^\mathsf{T} (A^\mathsf{T})^{T-t-1} W_T^\dagger (x_\mathrm{f} - A^T x(0)), \tag{4}$$

where $W_T^\dagger$ is the Moore-Penrose pseudoinverse of W_T [2].

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The controllability Gramian (3) and the minimum-energy control input (4) identify fundamental control limitations for the system (1), and have been extensively used to solve design [3], [4], sensor and actuator placement [5], [6], and control problems [7], [8] for systems and networks. However, besides their theoretical value, the optimal control input (4) is rarely used in practice or even computed numerically because (i) it relies on the perfect knowledge of the system dynamics, and its performance is not robust to model uncertainties, and (ii) the controllability Gramian is typically ill-conditioned, especially when the system is large [7], [9]. This implies that the control sequence (4) is numerically difficult to compute, and that its implementation leads to errors [10]. To the best of our knowledge, efficient and reliable methods to compute minimum-energy control inputs are still lacking.

Paper contributions. This paper features three main contributions. First, we show that minimum-energy control inputs for linear systems can be computed from data obtained from control experiments with non-minimum-energy inputs, and without knowledge or estimation of the system matrices. Thus, optimal inputs can be learned from non-optimal ones, and we provide three different expressions for doing so. Second, we provide bounds on the number of required control experiments as a function of the dimension of the system, number of control inputs, and length of the control horizon. Surprisingly, we show that a finite number of non-optimal control experiments is always sufficient to compute minimum-energy control inputs towards any reachable state. Third and finally, we show that the data-driven computation of minimum-energy inputs is numerically as reliable as the computation of the inputs based on the exact knowledge of the system matrices, and substantially more reliable than using the closed form expression based on the Gramian.

Our results suggest the tantalizing hypothesis that several optimal control problems can be solved efficiently and reliably using a combination of data-driven algorithms and system properties (in our setup, linearity of the dynamics), even when the system model is uncertain or unknown.

Related work. Several works investigate the problem of learning optimal controls for linear systems from input-output data. The classic model-based approach [11] consists of (i) identifying a model of the system from the available data, and (ii) using the estimated model to design the optimal control inputs. Data-driven algorithms have been proposed in [12]–[16] for the LQR/LQG problem. In particular, the approach pursued in these papers relies on the estimation of the Markov parameters of the system, thereby bypassing the identification step of the model-based approach. Differently from the above approaches, in this paper we focus on the problem of learning open-loop minimum-energy inputs from experimental data, without reconstructing the system parameters and where the experiments use arbitrary control inputs. To the best of our knowledge, this paper addresses a novel problem and provides entirely new and numerically more reliable expressions for the minimum-energy control inputs.

II. Learning minimum-energy control inputs

In vector form, the minimum-energy control problem asks to find the minimum-norm solution to the following equation:

$$x_{\mathrm{f}} = A^T x(0) + \underbrace{\begin{bmatrix} B & AB & \cdots & A^{T-1}B \end{bmatrix}}_{C_T} u,$$

where the vector $u \in \mathbb{R}^{mT}$ contains the control inputs over the control horizon $[0, T-1]$, and C_T denotes the T-steps controllability matrix.[1] Then, if the controllability matrix C_T is known, the minimum-energy control input to reach x_{f} is

$$u^* = C_T^{\dagger}(x_{\mathrm{f}} - A^T x(0)). \tag{5}$$

Instead, in this paper we aim to compute minimum-energy control inputs leveraging a set of control experiments and assuming that the system matrices, and thus the controllability matrix, are not available. In particular, we assume that a set of control experiments has been conducted, and that the resulting data (x_i, u_i), with $i \in \{1, \dots, N\}$, is available to estimate minimum-energy inputs, where

$$x_i = A^T x(0) + C_T u_i. \tag{6}$$

We remark that the inputs u_i are arbitrary and not necessarily of minimum-norm to reach the final state x_i. In vector form,

$$X = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix}, \text{ and } U = \begin{bmatrix} u_1 & \cdots & u_N \end{bmatrix}. \tag{7}$$

A. Data-driven minimum-energy control inputs

Because we only rely on the experimental data (X, U) to learn the minimum-energy control input to reach a desired state, we postulate that such input can be computed as a linear combination of the inputs U. Thus, we formulate and study the following constrained minimization problem:

$$\begin{aligned} \alpha^* = \quad & \arg\min_{\alpha} \quad \|U\alpha\|_2^2, \\ & \text{s.t.} \qquad x_{\mathrm{f}} = X\alpha. \end{aligned} \tag{8}$$

As we show in Theorem 2.1, a first data-driven expression for the minimum-energy control input derives from a solution to (8). We start with the expression of the minimum-energy control input for the case $x(0) = 0$, and we postpone the general case $x(0) \neq 0$ to Remark 3. Let $\mathrm{Im}(M)$ and $\mathrm{Ker}(M)$ denote the range-space and the null-space of the matrix M, respectively. With a slight abuse of notation, we write $K = \mathrm{Im}(A)$ (resp. $K = \mathrm{Ker}(A)$) to say that K is a basis of $\mathrm{Im}(A)$ (resp. $\mathrm{Ker}(A)$). A matrix is full row rank if the dimension of its range-space equals the number of its rows.

[1]To simplify the technical treatment and without compromising generality, we assume that x_{f} is reachable in T-steps, i.e. $(x_{\mathrm{f}} - A^T x(0)) \in \mathrm{Im}(C_T)$.

Theorem 2.1: ***(Data-driven minimum-energy control inputs when*** $x(0) = 0$***)*** If the matrix U in (7) is full row rank, then, for any final state x_{f}, the minimum-energy input equals

$$u^* = (I - UK(UK)^{\dagger})UX^{\dagger}x_{\mathrm{f}}, \tag{9}$$

where $K = \mathrm{Ker}(X)$ and X is as in (7).

Proof: Consider the minimization problem (9). Because U is full row rank, there exists α^* such that $u^* = U\alpha^*$, where u^* is the minimum-energy control input to reach x_{f}. Additionally, α^* satisfies the constraint in (8) because $X\alpha^* = C_T U\alpha^* = C_T u^* = x_{\mathrm{f}}$. Finally, because u^* is unique [1], α^* is also a solution to (8), and its computation is equivalent to computing the minimum-energy input u^*.

To compute α^*, we solve the constraint and substitute it in the cost function. Namely, $\alpha^* = X^{\dagger}x_{\mathrm{f}} + Kw$, where $K = \mathrm{Ker}(X)$ and w is an arbitrary vector. Equating to zero the derivative of the cost function with respect to w, we obtain

$$\alpha^* = X^{\dagger}x_{\mathrm{f}} - K\underbrace{(UK)^{\dagger}UX^{\dagger}x_{\mathrm{f}}}_{-w^*},$$

from which (9) follows by letting $u^* = U\alpha^*$. ■

Theorem 2.1 provides an expression of the minimum-energy control input, which only uses data originated from a set of control experiments, and does not require the knowledge of the system matrices. Importantly, Theorem 2.1 shows that minimum-energy control inputs can be directly computed based on a number of control experiments with arbitrary, thus not minimum-energy, inputs. Further, Theorem 2.1 assumes that U is full row rank, which guarantees the computation of the minimum-energy input for any final state x_{f}. When U is not full row rank but $u^* \in \mathrm{Im}(U)$, the minimum-energy control input can still be computed as in Theorem 2.1. Instead, when $u^* \notin \mathrm{Im}(U)$, the minimum-energy input cannot be computed as a (linear) combination of the experimental data (7). In this case, the data-driven control input (9) reaches the desired final state x_{f}, if $x_{\mathrm{f}} \in \mathrm{Im}(X)$, or the final state $\tilde{x}_{\mathrm{f}} \in \mathrm{Im}(X)$ that is closest to x_{f}, if $x_{\mathrm{f}} \notin \mathrm{Im}(X)$. To see this, let u^* be as in (9) and note that

$$\begin{aligned} \tilde{x}_{\mathrm{f}} &= C_T u^* = C_T(I - UK(UK)^{\dagger})UX^{\dagger}x_{\mathrm{f}} \\ &= C_T UX^{\dagger}x_{\mathrm{f}} - \underbrace{C_T UK(UK)^{\dagger}UX^{\dagger}x_{\mathrm{f}}}_{=0 \text{ because } C_T UK = XK = 0} = XX^{\dagger}x_{\mathrm{f}}, \end{aligned}$$

which shows that $\tilde{x}_{\mathrm{f}}$ is the orthogonal projection of x_{f} onto $\mathrm{Im}(X)$. This in particular implies that the error $\|x_{\mathrm{f}} - \tilde{x}_{\mathrm{f}}\|_2$ is non-increasing in the number of experiments N, and it vanishes when the experimental data satisfies $x_{\mathrm{f}} \in \mathrm{Im}(X)$. Finally, Theorem 2.1 can also be used to quantify the number of experiments needed to compute minimum-energy inputs.

Corollary 2.2: ***(Required number of control experiments to compute minimum-energy inputs)*** Let n be the dimension of the system, m the number of inputs, T the control horizon, and N the number of control experiments. Then,

(i) $N \geq n$ is necessary to compute minimum-energy control inputs towards any arbitrary final state x_{f};

(ii) $N = mT$ is sufficient to compute minimum-energy control inputs towards any arbitrary final state x_{f}, provided that the inputs u_i are linearly independent.

Proof: *(Necessity)* Assume by contradiction that the number of experiments is strictly less than n. Then, $\text{Rank}(X) < n$, and there exists $x_{\text{f}} \notin \text{Im}(X)$. Then, the minimization problem (8) is infeasible, and the minimum-energy control input cannot be computed from the inputs U.

(Sufficiency) Let the experimental inputs be linearly independent. Then, U is invertible and, for any x_{f}, there exists a solution α^* such that $u^* = U\alpha^*$. This shows that the minimum-energy input can be computed from the data. ■

Corollary 2.2 characterizes the number of control experiments that are required to compute minimum-energy control inputs from experimental data. In particular, as few as n experiments are needed, in which case the experiments must contain n linearly independent minimum-energy control inputs, and as many as mT experiments are sufficient, in which case the control inputs can be selected arbitrarily provided that they form a linearly independent set of vectors. This also shows that optimal control inputs can be learned from a *finite* number of non-optimal control inputs.

Remark 1: ***(Estimating the dimension of the system)*** It is interesting to note that the dimension n of the system can also be estimated from a finite number of control experiments. In fact, from Corollary 2.2 we know that the control input (9) reaches a randomly chosen final state x_{f} when the number of experiments satisfy $N = n$. Then, if n is unknown and $p > n$ measurements of the system state are available, the value n can be computed by iteratively trying the input (9) for different numbers of control experiments. This result is of general interest, and finds applicability beyond the considered control design problem. □

Remark 2: ***(Geometric properties of** (9)**)*** Several geometric properties of (9) can be highlighted. First, $UK = \text{Ker}(C_T)$ when U is full row rank. In fact, $C_T UK = XK = 0$, showing that $\text{Im}(UK) \subseteq \text{Ker}(C_T)$. Further, if $C_T u = 0$ and $u = U\alpha$, then, $X\alpha = C_T U\alpha = C_T u = 0$, showing that $\alpha \in \text{Im}(K)$ and $\text{Ker}(C_T) \subseteq \text{Im}(UK)$. Thus, $\text{Im}(UK) = \text{Ker}(C_T)$ when U is full row rank. Second, $(I - UK(UK)^\dagger)$ is the orthogonal projector onto the kernel of $(UK)^\mathsf{T}$ and, consequently, $u^* = (I - UK(UK)^\dagger)UX^\dagger x_{\text{f}}$ is orthogonal to $\text{Ker}(C_T)$. This is expected, because u^* is the minimum-energy control input to reach the state x_{f}. □

Remark 3: ***(Data-driven minimum-energy control inputs when** $x(0) \neq 0$**)*** When $x(0) \neq 0$, the computation of the minimum-energy control to reach x_{f} is more involved, as the unknown matrix A and vector $x(0)$ enter the relation (6). Yet, under a mild assumption on the experimental inputs U, minimum-energy inputs can still be computed with a finite number of experiments. To see this, consider the problem

$$\begin{aligned} \min_{\alpha} \quad & \|U\alpha\|_2^2, \\ \text{s.t.} \quad & x_{\text{f}} = X\alpha, \\ & 1 = \mathbb{1}^\mathsf{T}\alpha. \end{aligned} \tag{10}$$

Assume that the matrix U is full row rank, and that there exists a vector w such that $Uw = 0$ and $\mathbb{1}^\mathsf{T} w \neq 0$, where $\mathbb{1}$ denotes the vector of all ones.[2] Let

$$\alpha^* = U^\dagger u^* + \frac{1 - \mathbb{1}^\mathsf{T} U^\dagger u^*}{\mathbb{1}^\mathsf{T} w} w,$$

and notice that $u^* = U\alpha^*$, where u^* is the minimum-energy control input to reach x_{f}. Further, using (6) and $1 = \mathbb{1}^\mathsf{T}\alpha^*$,

$$\begin{aligned} X\alpha^* &= \sum_{i=1}^N X_i \alpha_i^* = A^T x(0) \sum_{i=1}^N \alpha_i^* + C_T \sum_{i=1}^N \alpha_i^* U_i \\ &= A^T x(0) + C_T u^* = x_{\text{f}}. \end{aligned}$$

Then, similarly to proof of Theorem 2.1, a solution to (10) determines the unique minimum-energy control input.

To solve the minimization problem (10), let $\bar{X} = [X^\mathsf{T} \ \mathbb{1}]^\mathsf{T}$ and $\bar{x}_{\text{f}} = [x_{\text{f}}^\mathsf{T} \ 1]^\mathsf{T}$. Then, similarly to Theorem 2.1, we obtain $\alpha^* = \bar{X}^\dagger x_{\text{f}} - K(UK)^\dagger U \bar{X}^\dagger \bar{x}_{\text{f}}$, where $K = \text{Ker}(\bar{X})$, and

$$u^* = (I - UK(UK)^\dagger)U\bar{X}^\dagger \bar{x}_{\text{f}}.$$

Finally, because the matrix U is required to have a nontrivial null-space, a sufficient number of linearly-independent non-optimal experiments for the computation of the minimum-energy control input to any arbitrary final state is $mT + 1$.□

B. Alternative derivations of minimum-energy controls

In this subsection we present different optimization problems that can be used to derive equivalent expressions of the data-driven minimum-energy control input (9). We start with the following minimization problem, which encodes the problem of estimating the controllability matrix from data:

$$C_T^* = \arg\min_C \quad \|X - CU\|_F^2, \tag{11}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. The above problem has a unique solution, which equals $C_T^* = XU^\dagger$. Notice that the minimization problem (11) returns an estimate of the controllability matrix, which can be used to compute the control as $\hat{u} = (C_T^*)^\dagger x_{\text{f}} = (XU^\dagger)^\dagger x_{\text{f}}$. We next show that $\hat{u}$ coincides with the control input (9).

Theorem 2.3: ***(Equivalent expressions of data-driven minimum-energy inputs)*** Let X and U be as in (7). Then,

$$(I - UK(UK)^\dagger)UX^\dagger x_{\text{f}} = (XU^\dagger)^\dagger x_{\text{f}}. \tag{12}$$

Proof: To prove the claim, we show that $(XU^\dagger)^\dagger = (I - UK(UK)^\dagger)UX^\dagger$. Let $K = I - X^\dagger X$. Since $I - UK(UK)^\dagger \triangleq P$ is the projection operator on $\text{Ker}((UK)^\mathsf{T})$,

$$(UK)^\mathsf{T} P = 0 \overset{P=P^\mathsf{T}}{\Longrightarrow} PUK = 0 \Rightarrow PUX^\dagger X = PU. \tag{13}$$

Because $X = C_T U$, we have $\text{Ker}(U) \subseteq \text{Ker}(X)$. Since $I - U^\dagger U$ is the projection operator on $\text{Ker}(U)$, we have

$$X(I - U^\dagger U) = 0 \Rightarrow XU^\dagger U = X. \tag{14}$$

Further, using $XK = 0$, we obtain

$$XU^\dagger(I - P) = XU^\dagger UK(UK)^\dagger \overset{(14)}{=} XK(UK)^\dagger = 0. \tag{15}$$

[2]These assumptions can always be satisfied by properly designing the experimental inputs, or by running sufficiently many random experiments.

Finally, since $I-UU^\dagger$ denotes the orthogonal projection operator on $\mathrm{Ker}(U^\mathsf{T})$ and $UK(UK)^\dagger$ the orthogonal projection operator on $\mathrm{Im}(UK) \subseteq \mathrm{Im}(U) \perp \mathrm{Ker}(U^\mathsf{T})$, we have

$$\begin{aligned}&\underbrace{UK(UK)^\dagger}_{=I-P}(I-UU^\dagger)=0\\&\Rightarrow (I-P)(I-UU^\dagger)=[(I-P)(I-UU^\dagger)]^\mathsf{T}\\&\overset{(a)}{\Rightarrow} UU^\dagger P = PUU^\dagger,\end{aligned} \tag{16}$$

where (a) follows because $I-P$ and $I-UU^\dagger$ are symmetric.

To conclude the proof, we show that $PUX^\dagger$ is the pseudoinverse of $XU^\dagger$ by proving the Moore-Penrose conditions:

(i) $PUX^\dagger XU^\dagger PUX^\dagger \overset{(13)}{=} PUU^\dagger PUX^\dagger \overset{(16)}{=} P^2UU^\dagger UX^\dagger = PUX^\dagger$;

(ii) $XU^\dagger PUX^\dagger XU^\dagger \overset{(13)}{=} XU^\dagger PUU^\dagger = XU^\dagger UU^\dagger - XU^\dagger(I-P)UU^\dagger \overset{(15)}{=} XU^\dagger$;

(iii) $XU^\dagger PUX^\dagger = XU^\dagger UX^\dagger - XU^\dagger(I-P)UX^\dagger \overset{(14),(15)}{=} XX^\dagger = (XX^\dagger)^\mathsf{T}$;

(iv) $PUX^\dagger XU^\dagger \overset{(13)}{=} PUU^\dagger \overset{(16)}{=} UU^\dagger P = (PUU^\dagger)^\mathsf{T}$.

■

The minimization problem (11) reconstructs the forward controllability matrix C_T, from which minimum-energy control inputs can be derived by subsequently computing $C_T^\dagger$. To avoid the computation of $C_T^\dagger$ and obtain a potentially simpler expression, we next consider the problem of directly estimating $C_T^\dagger$ from the experimental data:

$$M^* = \arg\min_M \quad \|MX-U\|_F^2. \tag{17}$$

Notice that the latter problem is equivalent to estimating the inverse map from X to U, and it is typically more difficult than the problem of estimating the map from U to X. In fact, while the forward map is unique, the inverse map is typically not.[3] Further, the control input M^*x_f obtained by solving the minimization problem (17) is not guaranteed to be of minimum norm and to steer the system to x_f, as these constraints do not appear in the minimization problem. In what follows, we say that a sequence of random matrices $\{X_n\}_{n\in\mathbb{N}}$ converges almost surely (a.s.) to a matrix X, and denote it with $X_n \xrightarrow{\text{a.s.}} X$, if $\Pr(\lim_{n\to\infty} X_n = X)=1$.

Theorem 2.4: ***(Asymptotically equivalent expression to*** (9)***)*** Let X and U be as in (7). The unique solution to the minimization problem (17) is

$$M^* = UX^\dagger, \tag{18}$$

and the corresponding control input can be written as

$$\hat{u} = M^*x_\mathrm{f} = UX^\dagger x_\mathrm{f}. \tag{19}$$

Further, if X is full row rank, then $C_TM^*x_\mathrm{f} = x_\mathrm{f}$. That is, the control $\hat{u}$ steers the system from $x(0)=0$ to $x(T) = x_\mathrm{f}$. Finally, if the entries of U are i.i.d. random variables with zero mean and nonzero finite variance, then

$$UX^\dagger \xrightarrow{\text{a.s.}} C_T^\dagger \quad \text{as } N\to\infty.$$

[3]In particular, the inverse map is not unique whenever $mT > n$.

That is, as the number of control experiments increases, the input $\hat{u}$ converges almost surely to the input u^* in (9).

Proof: The expression (18) follows from the properties of the Moore-Penrose pseudoinverse. For the second claim,

$$C_T\hat{u} = C_TUX^\dagger x_\mathrm{f} = XX^\dagger x_\mathrm{f} = x_\mathrm{f},$$

where we have used that X is full row rank and $X = C_TU$. To prove the third statement, let $N\to\infty$, and let the control experiments be chosen so that the entries of U are i.i.d. random variables with zero mean and finite variance σ^2. Let U_{ij} denote the (i,j)-th entry of U, and observe that the (i,j)-th entry of $\frac{1}{N}UU^\mathsf{T}$ equals $\frac{1}{N}\sum_{k=1}^N U_{ik}U_{jk}$. Because $\{U_{ik}U_{jk}\}_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables, for all $i,\, j\in\{1,\dots,N\}$ and, due to the Strong Law of Large Numbers [17, p. 6], when $N\to\infty$ we have

$$\frac{1}{N}\sum_{k=1}^N U_{ik}U_{jk} \xrightarrow{\text{a.s.}} \mathbb{E}[U_{i1}U_{j1}] = \begin{cases}\sigma^2, & \text{if } i=j,\\ 0, & \text{if } i\neq j,\end{cases}$$

where $\mathbb{E}[\cdot]$ denotes the expected value operator. Then,

$$\frac{1}{N}UU^\mathsf{T} \xrightarrow{\text{a.s.}} \sigma^2 I \quad \text{as } N\to\infty. \tag{20}$$

Next, consider the function $f:\mathbb{R}^{mT\times mT}\to\mathbb{R}^{mT\times n}$

$$f(Y) = YC_T^\mathsf{T}(C_TYC_T^\mathsf{T})^\dagger.$$

Note that $f(Y)$ is continuous at $Y=\alpha I$, $\alpha>0$,[4] and $f(\alpha I) = C_T^\mathsf{T}(C_TC_T^\mathsf{T})^\dagger = C_T^\dagger$ [2, p. 49]. To conclude the proof, we employ the Continuous Mapping Theorem [17, Theorem 2.3(iii)] and (20) to obtain, as $N\to\infty$,

$$\begin{aligned}UX^\dagger = U(C_TU)^\dagger &= \frac{1}{N}UU^\mathsf{T}C_T^\mathsf{T}\left(C_T\frac{1}{N}UU^\mathsf{T}C_T^\mathsf{T}\right)^\dagger\\ &= f\left(\frac{1}{N}UU^\mathsf{T}\right) \xrightarrow{\text{a.s.}} f\left(\sigma^2 I\right) = C_T^\dagger.\end{aligned}$$

■

Theorem 2.4 contains a data-driven expression of the minimum-energy control input for a linear system, which does not rely on the estimation of the system matrices or the controllability matrix. As we show in the next section, the expression (19) is not only conceptually simpler than the classic Gramian-based expression of the minimum-energy control input and our other data-driven expressions (9) and (12), but it is also numerically more reliable as it requires a smaller number of operations. Yet, differently from (9) and (12), the expression (19) coincides with the minimum-energy control only asymptotically in the number of experiments.

Remark 4: ***(Equivalent expressions of data-driven minimum-energy control inputs when*** $x(0)\neq 0$***)*** Following Remark 3 and Theorem 2.3, the data-driven minimum-energy input to steer the system from the (nonzero) initial state $x(0)$ to the final state x_f can be written equivalently as

$$u^* = (I-UK(UK)^\dagger)U\bar{X}^\dagger\bar{x}_\mathrm{f} = (\bar{X}U^\dagger)^\dagger\bar{x}_\mathrm{f} = U\bar{X}^\dagger\bar{x}_\mathrm{f},$$

where U, X are as in (7), $\bar{X} = [X^\mathsf{T}\ \mathbb{1}]^\mathsf{T}$, $\bar{x}_\mathrm{f} = [x_\mathrm{f}^\mathsf{T}\ 1]^\mathsf{T}$, and the last equality holds asymptotically as in Theorem 2.4. □

[4]In fact, since $\mathrm{Rank}(C_TYC_T^\mathsf{T}) = \mathrm{Rank}(C_TC_T^\mathsf{T})$ for any positive definite Y, it holds $\lim_{k\to\infty}(C_TY_kC_T^\mathsf{T})^\dagger = (\alpha C_TC_T^\mathsf{T})^\dagger$ for any sequence of positive definite matrices $\{Y_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} Y_k = \alpha I$ [2, p. 238].

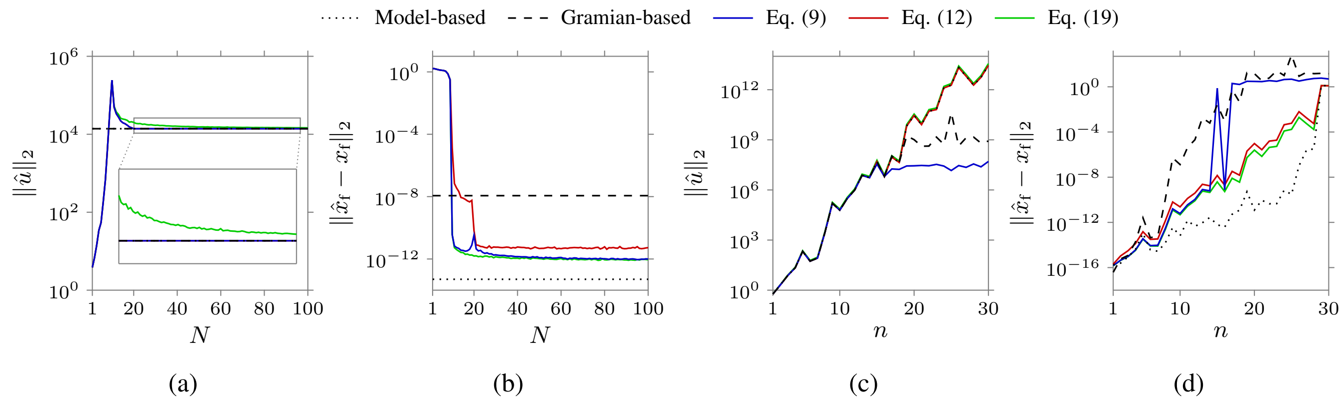

Fig. 1. Fig. 1(a)-(b) show the norm of the control input $\hat{u}$ computed via the model-based formula (5) (dotted line), via inversion of the controllability Gramian (dashed line), and via the data-driven expressions in Eq. (9), (12), and (19) (colored lines), and the corresponding error in the final state, as the number of data N varies. We choose $n = 10$, $m = 1$, and $T = 20$. The matrix A has been populated with random i.i.d. normal entries and then normalized to ensure stability, $B = [1\ 0\ \cdots\ 0]^\mathsf{T}$, and the entries of x_f have been chosen randomly according to a normal distribution. The curves represent the average over 100 experiments with data pairs (x_i, u_i), where u_i has random i.i.d. normal entries, and $x_i = C_T u_i$. While the inputs (9) and (12) have minimum-energy after a finite number of experiments, as predicted by Theorems 2.1 and 2.3 and shown in Fig. 1(a), from the zoom-in box of Fig. 1(a) we note that the norm of the control input (19) approaches its minimum value only as N grows, as predicted by Theorem 2.4. Fig. 1(c)-(d) show the norm of the control inputs $\hat{u}$ computed as above, and the corresponding errors in the final state, as a function of the system dimension n. We choose $m = 1$, $T = 2n$, and $N = mT + 20$. The matrices A, B, and the state x_f have been generated as above. The curves represent the average over 1000 experiments with data pairs (x_i, u_i) generated as above. All the computations have been carried out in Matlab 2016b using standard built-in linear algebra routines.

C. *Numerical analysis*

What remains unclear from the previous analysis is the benefit, if any, in collecting a large number of control experiments. We next show that increasing the number of control experiments can improve the numerical reliability and accuracy of computing minimum-energy control inputs. For a fair comparison, we use the built-in Matlab functions for all computations, and remark that better numerical performance may be obtained using more accurate mathematical routines.

In Fig. 1 we compare the numerical performance of the model-based expressions of the minimum-energy control $u^* = C_T^\dagger x_\mathrm{f}$ and $u^* = C_T^\mathsf{T} W_T^\dagger x_\mathrm{f}$ (Gramian-based), with our data-driven expressions in (9), (12), and (19). In particular, in Fig. 1(a)-(b) we plot the norm of the control inputs and the numerical errors in reaching the final state x_f, for all strategies and as a function of the number N of control experiments. Fig. 1(a) shows that the norm of the data-driven control inputs (9) and (12) equals its minimum value when $N \geq mT$ (as predicted by Theorems 2.1 and 2.3), whereas the norm of the data-driven input (19) converges to its minimum value only asymptotically (as predicted by Theorem 2.4). Fig. 1(b) shows that, for sufficiently large N, the final state reached by the three data-driven control strategies is almost as close to x_f as the one computed via the model-based formula $u^* = C_T^\dagger x_\mathrm{f}$, and considerably closer to x_f than the state reached by the Gramian-based control input, with expressions (9) and (19) being the most accurate, showing that the computation of the minimum-energy control input via our data-driven expression is as reliable as the computation of the input based on the exact knowledge of the system matrices, and numerically more reliable than the model-based Gramian formula. Instead, in Fig. 1(c)-(d) we plot the norm of the control inputs obtained through the different strategies described above and their corresponding errors in the final state as a function of the system dimension n. As expected, the accuracy of the Gramian-based control input deteriorates rapidly as n increases. Yet, surprisingly, the data-driven expressions of the minimum-energy control inputs remain accurate for systems of considerably larger dimension. Further, the data-driven control (19) yields the smallest error in the final state among the three data-driven strategies. This could be due to the simpler form of (19), which requires the computation of only one pseudoinverse, or to the fact that the energy of (19) reaches the minimum value only asymptotically in N. Finally, Fig. 1(c)-(d) show that expression (9) becomes numerically unreliable for smaller values of the system dimension compared to (12) and (19). This is likely because of the additional computations in (9).

In Fig. 2 we numerically investigate the performance of different neural networks in learning minimum-energy control inputs from the experimental data X and U. We use a dynamical system of dimension $n = 2$ with 1 control input, and a control horizon $T = 4$. In Fig. 2(a)-(b) we plot the norm of the control input learned by the neural network and the corresponding error in the final state as a function of the number of data N, for three different number of hidden layers and a fixed activation function (hyperbolic tangent sigmoid function). In Fig. 2(c)-(d), we repeat the same experiments for a fixed number of hidden layers and three different choices of activation functions (hyperbolic tangent sigmoid, log-sigmoid, and saturating linear functions). Our results show that, despite the simplicity of the considered minimum-energy control task, the inputs obtained by the considered neural networks are not of minimum-norm, and in fact they also fail to steer the state of the system to the desired final state. These results should be interpreted as a critical

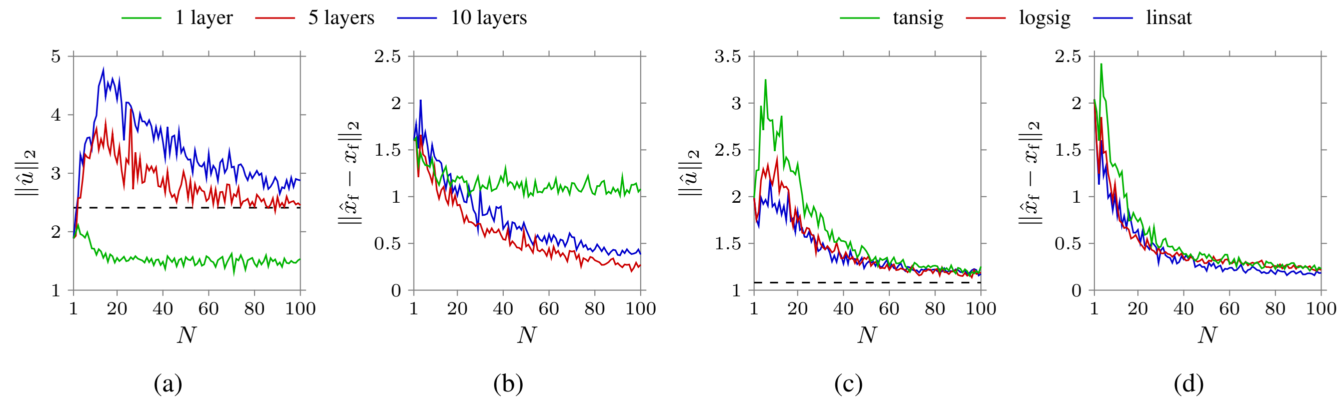

Fig. 2. Fig. 2(a)-(b) show the norm of the control input $\hat{u}$ obtained as the output of a neural network with hyperbolic tangent sigmoid activation function, and the corresponding error in the final state, for three different numbers of hidden layers. Fig. 2(c)-(d) show the norm of the control input $\hat{u}$ obtained as the output of a neural network with ten hidden layers, and the corresponding error in the final state, for three different choices of activation function, namely, hyperbolic tangent sigmoid (tansig), log-sigmoid (logsig), and saturating linear (linsat). In all plots, we set $n = 2$, $m = 1$, and $T = 4$, and we used the Matlab built-in functions `fitnet` and `train` (Deep Learning Toolbox) to create the above neural networks and train them with input-output data (X, U). We divided the data into approximately 70% training, 15% validation, and 15% testing, and we used Levenberg–Marquardt backpropagation for training. The matrix A has been generated with random i.i.d. normal entries and then normalized to ensure stability, $B = [1\,0]^{\mathsf{T}}$, and the entries of x_{f} have been chosen randomly according to a normal distribution. In the plots, the solid curves represent the average over 100 experiments with data pairs (x_i, u_i), where u_i has random i.i.d. normal entries, and $x_i = C_T u_i$. The dashed black lines in Fig. 2(a)-(c) represent the norm of the minimum-energy inputs.

warning in blindly using machine learning techniques for solving control problems, and they strengthen our combined approach to obtain the controls (9), (12), and (19), which leverages both experimental data and system properties.

III. Conclusion and future work

In this paper we pursue a combined data-driven and model-based approach to compute open-loop minimum-energy control inputs for linear systems. Leveraging linearity of the dynamics, we show that such optimal controls can be learned from a finite number of control experiments, without knowing or reconstructing the system matrices, and where the control experiments are conducted with non-optimal and arbitrary inputs. We further illustrate that, surprisingly, our data-driven expressions of the minimum-energy control inputs are simpler and numerically more reliable than the classic Gramian-based expression of the open-loop minimum-energy control input, especially when the dimension of the system increases. Finally, we investigate the effectiveness of standard neural networks in reconstructing minimum-energy control inputs from non-optimal data, and show that the inputs obtained with these methods are not optimal and even fail at steering the system state towards the desired final state.

The results of this paper support the intriguing idea of combining model-based control methods with data-driven techniques, showing that this new control framework has the potential to considerably increase the reliability and effectiveness of the two parts alone. This paper also creates several directions of future research, including the extension to closed-loop optimal control problems and the investigation of data-driven network control, and it promotes a rigorous approach for the design of data-driven control algorithms.

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