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Networks with diagonal controllability Gramian: Analysis, graphical conditions, and design algorithms ☆

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ABSTRACT

This paper aims to establish explicit relationships between the controllability degree of a network, that is, the control energy required to move the network between different states, and its graphical structure and edge weights. As it is extremely challenging to accomplish this task for general networks, we focus on the case where the network controllability Gramian is a diagonal matrix. The main technical contributions of the paper are (i) to derive necessary and sufficient graphical conditions for networks to feature a diagonal controllability Gramian, and (ii) to propose a constructive algorithm to design network topologies and weights so as to generate stable and controllable networks with pre-specified diagonal Gramians. The proposed network design algorithm allows for individual assignment of how each node responds to external stimuli, so as to selectively enforce robustness to external disturbances. While relying on the simplifying assumption of a diagonal controllability Gramian, our analysis reveals novel and counterintuitive controllability properties of complex networks. For instance, we identify a class of continuous-time networks where the control energy is independent of their cardinality and number of control nodes (thus disproving existing results based on numerical controllability studies), discuss their stability margin, and show that the energy required to control a node can be made independent of its graphical distance from the control nodes. These results complement and formally support, or challenge, a series of conjectures based on numerical studies in the field of complex networks.

1. Introduction

Real-world systems, including interconnected power, social, and cyber–physical systems, can often be represented as complex networks, and their dynamic behavior depends to a large extent upon the properties of their interconnection structure. This relationship between the behavior of a complex system and the properties of its abstract graphical representation has motivated numerous studies over the last decade across different research communities, which aim to identify which network features are mostly responsible for desirable dynamic properties such as efficiency, robustness, and controllability.

In this paper we unveil novel relationships between the structure of a network and its quantitative controllability properties, and we propose an algorithm to construct networks with pre-specified controllability properties (a previously unsolved network design problem). Controllability indicates whether the state of a network can be arbitrarily changed by a suitable choice of exogenous inputs (Chen, Lin, & Shamash, 2004; Kalman, Ho, & Narendra, 1963). While classic results guarantee that network controllability depends (in a generic sense (Reinschke, 1988)) uniquely on the network structure and is independent of the network weights (Aguilar & Gharesifard, 2015; Chapman & Mesbahi, 2013a; Ji, Lin, & Yu, 2015; Lin, 1974; Liu, Slotine, & Barabási, 2011; Lou & Hong, 2012; Olshevsky, 2014; Parlangeli & Notarstefano, 2012; Pequito, Ramos, Kar, Aguiar, & Ramos, 2014; Rahmani, Ji, Mesbahi, & Egerstedt, 2009; Zhang, Cao, & Camlibel, 2014), the degree of controllability, that is, the energetic effort required to control the state to arbitrary configurations, is an intricate function of all network parameters (Bof, Baggio, & Zampieri, 2017; Dhal & Roy, 2016; Kumar, Menolascino, Kafashan, & Ching, 2015; Pasqualetti, Zampieri, Bullo, & metrics, 2014; Summers, Cortesi, & Lygeros, 2016; Tzoumas, Rahimian, Pappas, & Jadbabaie, 2016; Yan, Ren, Lai, Lai, & Li, 2012; Yan, Tsekenis, Barzel, Slotine, Liu, & Barabási, 2015; Zelazo & Mesbahi, 2011; Zhao & Cortés, 2016). The controllability degree of a network, which is the notion investigated in this paper, has practical implications for the control of large networks, because several

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networks tend to become difficult to control when their control energy grows with the network cardinality, e.g., see Pasqualetti et al. (2014). Further, differently from its binary counterpart, the controllability degree can also be used as a measure of robustness and security. For example, if changing the state of a node requires large energy, then such node is resilient against low-energetic inputs representing disturbances or attacks.

The objectives of this paper are to derive analytical relationships between the controllability degree of a network and its graphical structure, and to design networks with pre-specified controllability degrees. Such analytical relations are particularly important because the numerical investigation of network controllability often leads to ill-conditioned problems (see also Remark 1). Although recent studies have connected the controllability degree of a network to its centrality (Bof et al., 2017), weights distribution (Pasqualetti & Zampieri, 2014), and geometry (Pasqualetti et al., 2014; Yan et al., 2015), precise graphical interpretations of the network controllability degree are still critically lacking and difficult to obtain in the general case. To overcome these difficulties and achieve our objectives, we restrict our analysis to networks featuring a diagonal controllability Gramian (Kailath, 1980), where the energy required to control each individual nodes is simply given by the diagonal entries of the Gramian. Our choice is not only convenient for the analysis and design of networks, but it also leads to important results. First, systems featuring diagonal Gramian matrices have been studied in the context of diagonal stability (Geromel, 1985; Hershkowitz, 1992; Kaszkurewicz & Bhaya, 2000; Kaszkurewicz & Hsu, 1984). We extend these results by solving the case where the right-hand side of the Lyapunov equation is only negative semi-definite (number of inputs smaller than the number of states as in a practical network system), and by deriving graphical conditions instead of only algebraic relations. Second, as we further articulate in Remark 1 below, our analysis leads to novel results in the context of control of complex networks with general topologies, thus justifying the use of a simplified setting. Third and finally, our method allows us to derive a first solution (to the best of our knowledge) to the problem of designing sparse systems with prescribed controllability properties. Relevant works include (Chapman & Mesbahi, 2013b; Chapman, Schoof, & Mesbahi, 2013; Gasparri, Pasqualetti, Santini, & Panzieri, 2016; Wan, Roy, & Saberi, 2007) where, however, the considered cost function is the $\mathcal{H}_2$ norm of the network system, rather than the network controllability degree or the $\mathcal{H}_2$ norm at the individual nodes, as we do.

The main technical contributions of this paper are as follows. First, we derive necessary and sufficient conditions for a network to feature a diagonal controllability Gramian (Theorem 3.1). In particular we show that, for the controllability Gramian to be diagonal, the network must be *sign-skew-symmetric*, that is, the weights of the two directional edges (i, j) and (j, i) between any two nodes i and j must have opposite sign, and *uniformly input-connected*, that is, for every node i and control node j, all paths from j to i must have the same weight ratio product. Second, we characterize stability and controllability of the set of networks with diagonal controllability Gramian (Theorem 4.2). Specifically we show that, for every positive definite diagonal matrix W and set of control nodes $\mathcal{S}$, there exists a dense set of stable networks that are controllable from $\mathcal{S}$ and have Gramian W. Third, we propose an algorithm to construct networks with desired diagonal Gramian (Algorithm 1). Because the Gramian determines the $\mathcal{H}_2$ norm of a system (Skogestad & Postlethwaite, 2005), our algorithm can be used not only to construct networks with desired controllability properties, but also to individually assign the robustness and security of each node against exogenous disturbances.

In addition to their technical contributions, the results presented in this paper have important implications beyond the considered scenario. For instance, by extending the results in Pasqualetti and Zampieri (2014), Pasqualetti et al. (2014) and Bianchin, Pasqualetti, and Zampieri (2015), we show that (i) there exists a class of networks whose control energy is independent of the cardinality of the network and the number of control nodes. This is a counterexample for the (numerical) conclusion in Yan et al. (2015) that the control energy of dynamical networks increases with its cardinality. (ii) The energy required to change the state of a node may be independent of its graphical distance to the control nodes. Nodes that are located close to the control nodes may require high control energy while nodes that are far away from the control nodes may require low control energy. (iii) The controllability matrix and controllability Gramian are usually treated as two equivalent tools to assess controllability. Yet, we show via examples that the two matrices may give extremely different, even opposite, quantitative measures of the controllability degree of a network. Finally, (iv) the controllability degree of a network, which depends on both structure and weights, can be used as a design criteria for security applications. Other implications of our results are discussed in Remark 1.

The rest of the paper is organized as follows. Section 2 contains our problem setup and preliminary notions. Our conditions for a diagonal network controllability Gramian are described in Section 3. Section 4 contains our results for the stability and controllability of the set of networks with diagonal controllability Gramian. Our network design algorithm and numerical examples are reported in Sections 5 and 6, respectively. Section 7 concludes the paper.

2. Problem statement and preliminary notions

Consider a network with n nodes and represented by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively. Let $\mathcal{V}_c = \{k_1, \ldots, k_{n_c}\} \subseteq \mathcal{V}$ be the set of control nodes, which receive n_c independent external control inputs. Following a large body of literature for the study of complex network dynamics, e.g., see Liu et al. (2011), Yan et al. (2012) and Allesina, Pascual, structure, and modules (2008), we let the network evolve with linear time-invariant dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\mathrm{T} \in \mathbb{R}^n$ contains the states of the nodes at time $t \in \mathbb{R}_{\geq 0}$, and $u(t) \in \mathbb{R}^{n_c}$ is the input vector. The matrix $A = [a_{ij}]$ is the weighted adjacency matrix of the graph $\mathcal{G}$, where $a_{ij} \neq 0$ when there is a directed edge from node j to node i. Two nodes are adjacent if either $a_{ij} \neq 0$ or $a_{ji} \neq 0$. Let I_n be the $n \times n$ identity matrix, and let e_i denote its ith column. Then, the input matrix B reads as

$$B = [\mathrm{e}_{k_1}, \mathrm{e}_{k_2}, \ldots, \mathrm{e}_{k_{n_c}}] \in \mathbb{R}^{n \times n_c}. \tag{2}$$

The input matrix B indicates the number and locations of the control nodes. In particular, Eq. (2) implies that BB^T is a diagonal matrix, where the ith diagonal entry equals 1 if $i \in \mathcal{V}_c$ and 0 otherwise.[1] In this paper we assume that B is given, and find conditions on the adjacency matrix A so that the network (1) is stable and controllable with diagonal and pre-specified Gramian.

The system (1) or, equivalently, the pair (A, B) is controllable if and only if the controllability matrix

$$K := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n \times (nn_c)} \tag{3}$$

has full row rank (Kalman et al., 1963). Controllability of (A, B) can also be assessed via the controllability Gramian, defined as

$$W_{t_f} := \int_0^{t_f} e^{At}BB^\mathrm{T}e^{A^\mathrm{T}t}\mathrm{d}t, \quad t_f > 0. \tag{4}$$

[1] The results extend directly to the case where the nonzero entries of B are different from 1. However, the case of non-dedicated control inputs, i.e., when the columns of B have multiple nonzero entries, requires additional investigation.

The controllability Gramian is an $n \times n$ positive semi-definite matrix, and it becomes positive definite if and only if the system is controllable (Chen et al., 2004, Theorem 3.4.1).

The Gramian can be used to calculate the minimum energy required to control the state between two arbitrary values. In particular, let x_f be the desired final state of the system. The control input $u(t) = B^{\mathrm{T}} e^{-A^{\mathrm{T}}(t-t_f)} W_{t_f}^{-1} x_f$ achieves the state transfer $x(t_f) = x_f$ from $x(0) = 0$, with minimum energy (Chen et al., 2004)

$$\int_0^{t_f} u^{\mathrm{T}}(t)u(t)\mathrm{d}t = x_f^{\mathrm{T}} W_{t_f}^{-1} x_f. \tag{5}$$

A larger t_f leads to less control energy for the same x_f, because $W_{t_f'} > W_{t_f}$ for $t_f' > t_f$. When $t_f = \infty$, we have

$$W := W_\infty = \int_0^{\infty} e^{At} BB^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathrm{d}t. \tag{6}$$

For unstable systems, (5) may diverge and W is ill-defined. For stable systems, W is well-defined and equals the unique solution to the Lyapunov equation (Chen et al., 2004)

$$AW + WA^{\mathrm{T}} = -BB^{\mathrm{T}}. \tag{7}$$

The minimum energy to control the network from the origin to some state x_f is $x_f^{\mathrm{T}} W^{-1} x_f$. By setting $x_f = \mathrm{e}_i$, with $i \in \{1, \ldots, n\}$, we define the ith *nodal energy* as

$$\varepsilon_i := \mathrm{e}_i^{\mathrm{T}} W^{-1} \mathrm{e}_i = [W^{-1}]_{ii},$$

where $[W^{-1}]_{ii}$ denotes the ith diagonal entry of W^{-1}. Intuitively, the ith nodal energy measures the energy needed to change the state of the ith node from 0 to 1, while keeping the final state of the remaining nodes at 0. Compared to the classic notion of *eigen energy* (Yan et al., 2015), where the final state x_f is aligned with the eigenvectors of the Gramian, our notion of nodal energy provides a more direct measure of the energy needed to control the state of the individual nodes of the network. Clearly, when W is diagonal with entries $\{\lambda_1, \ldots, \lambda_n\}$, nodal- and eigen-energies coincide, and $\varepsilon_i = \lambda_i^{-1}$. Finally, our notion of nodal energy can also be used to measure the robustness of each node against external disturbances. To see this, let $G(s)$ be the transfer function of the system $(A, B, \mathrm{e}_i^{\mathrm{T}})$. The $\mathcal{H}_2$ norm of the system satisfies (Skogestad & Postlethwaite, 2005, Section 4.10.1)

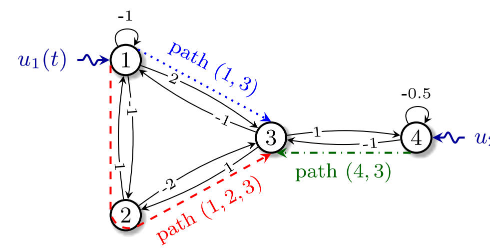
$$\begin{aligned} \|G\|_2^2 &= \frac{1}{2\pi} \mathrm{tr} \int_{-\infty}^{\infty} G(j\omega)G^*(j\omega)\mathrm{d}\omega \\ &= \mathrm{tr}(\mathrm{e}_i^{\mathrm{T}} W \mathrm{e}_i) = [W]_{ii} = [W^{-1}]_{ii}^{-1} = \varepsilon_i^{-1}, \end{aligned} \tag{8}$$

where the last equality holds when W is diagonal. Recall that the $\mathcal{H}_2$ norm of a system equals the expected root mean square of the output in response to white noise excitation or, equivalently, the energy of the output response to unit impulsive inputs (Skogestad & Postlethwaite, 2005). We conclude that a larger nodal energy implies stronger robustness of the node against input disturbances.

3. Graphical conditions for diagonally admissible networks

In this section we derive necessary and sufficient conditions for a network to feature a diagonal Gramian. Since the Gramian of a stable network is the solution to the Lyapunov equation (7), we characterize under what condition such equation admits diagonal positive-definite solutions. We start with the following definition.

Definition 1 (*Diagonally Admissible Network*)**.** Let W be positive definite and diagonal, and let B be as in (2). A network with adjacency matrix A is *diagonally admissible* for W and B if $AW + WA^{\mathrm{T}} = -BB^{\mathrm{T}}$. Further, let $\mathcal{D}(W, B) := \{A : AW + WA^{\mathrm{T}} = -BB^{\mathrm{T}}\}$ be the set of all diagonally admissible networks for W and B. □



$$A = \begin{bmatrix} -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & -1 \\ 0 & 0 & 1 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 1. The network is sign-skew-symmetric and its Gramian is diagonal. For node 3, there are three paths from the two inputs: (1, 3), (1, 2, 3), and (4, 3). The control impacts along the three paths are $\beta_{13} = \beta_{123} = \beta_{43} = 1$.

It should be noticed that the set $\mathcal{D}(W, B)$ contains both stable and unstable adjacency matrices A. When A is stable, W is the unique solution to the Lyapunov equation (7) and, thus, it equals the network controllability Gramian. When A is unstable, the Lyapunov equation (7) may admit multiple solutions, one of which is W. In this section we derive necessary and sufficient conditions for a network to be diagonally admissible. In the next section we will further characterize stability and controllability of diagonally admissible networks.

Because $W = \mathrm{diag}(w_1, \ldots, w_n) > 0$ and $[BB^T]_{ii} = 1$ if $i \in \mathcal{V}_c$ ($[BB^T]_{ii} = 0$ otherwise), we have $[AW + WA^{\mathrm{T}}]_{ii} = 2a_{ii}w_i$ and $[AW + WA^{\mathrm{T}}]_{ij} = a_{ij}w_j + w_i a_{ji}$. By equating the diagonal and off-diagonal entries of both sides of the Lyapunov equation (7), we obtain the following necessary and sufficient algebraic conditions for the Lyapunov equation (7) to admit a diagonal solution W:

$$[AW + WA^{\mathrm{T}}]_{ii} = 2a_{ii}w_i = \begin{cases} -1, & i \in \mathcal{V}_c, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

$$[AW + WA^{\mathrm{T}}]_{ij} = a_{ij}w_j + w_i a_{ji} = 0, \quad i \neq j. \tag{10}$$

We next elaborate on equations (9)–(10) to obtain useful graphical insights. We first define two important notions.

Definition 2 (*Sign-Skew-Symmetric Network*)**.** A network is *sign-skew-symmetric* if its adjacency matrix satisfies

(i) $a_{ii} < 0$ if $i \in \mathcal{V}_c$ and $a_{ii} = 0$ otherwise;
(ii) $a_{ij}a_{ji} < 0$ if nodes i and j are adjacent, and $a_{ij} = a_{ji} = 0$ otherwise. □

Condition (i) requires the control nodes to have negative self-loops, and the remaining nodes to have no self-loops. Condition (ii) requires adjacent nodes to be connected by two edges with opposite signs and directions. See Fig. 1 for an example of sign-skew-symmetric network.

It can be verified that any network satisfying (9)–(10) is sign-skew-symmetric. Yet, the converse is not true. In fact, in order to satisfy conditions (9) and (10), a sign-skew-symmetric network must satisfy additional constraints on the *control impacts*, which we define below.

Definition 3 (*Control Impact Along a Path*)**.** Given a sign-skew-symmetric network, consider the path $(i_1, i_2, \ldots, i_p)$ with $i_1 \in \mathcal{V}_c$ and $i_p \in \mathcal{V}$. The *control impact* on node i_p along the path $(i_1, i_2, \ldots, i_p)$ is

$$\beta_{i_1 \ldots i_p} := \frac{1}{|2a_{i_1 i_1}|} r_{i_1 \ldots i_p}, \tag{11}$$

where

$$r_{i_1\dots i_p} := \left|\frac{a_{i_2 i_1}}{a_{i_1 i_2}}\right| \left|\frac{a_{i_3 i_2}}{a_{i_2 i_3}}\right| \cdots \left|\frac{a_{i_p i_{p-1}}}{a_{i_{p-1} i_p}}\right|$$

is the *weight ratio product* along the path. In the case of $p = 1$, we let $\beta_{i_1 i_1} := 1/|2a_{i_1 i_1}|$. $\square$

The control impact in Definition 3 may be interpreted as the influence of a control input on a node in a network. Since there may be multiple paths from a control node to a given node, an input may have different impacts on the same node along different paths. When all inputs have the same control impact on every node along all different paths, the network is called *uniformly input-connected*.

Definition 4 (*Uniformly Input-Connected Networks*)**.** A network is *uniformly input-connected* if (i) it is sign-skew-symmetric, and (ii) for every node $i \in \mathcal{V}$ and $j \in \mathcal{V}_c$, the control impact along every path $(j, \dots, i)$ equals β_i, where β_i is a constant that depends only on i. $\square$

To illustrate Definition 4, consider the network in Fig. 1. For node 3, for instance, there are three different paths from the two inputs: $(1, 3), (1, 2, 3)$, and $(4, 3)$. It can be verified that the control impacts along the three paths are all equal to 1. Further, because (i) in Definition 4 is also satisfied, this network is uniformly input-connected.

We are now ready to state the main result of this section.

Theorem 3.1 (*Graphical Condition for Diagonally Admissible Networks*)**.** *A network is diagonally admissible if and only if it is uniformly input-connected. When a network is uniformly input-connected, it is diagonally admissible for*

$$W = \mathrm{diag}(\beta_1, \dots, \beta_n), \tag{12}$$

where $\beta_i > 0$ is the control impact of node i.

Proof. (Necessity) Suppose a network is diagonally admissible for $W = \mathrm{diag}(w_1, \dots, w_n)$. Then, Eqs. (9) and (10) hold. We only need to show that the two conditions in Definition 4 are satisfied. First, it can be easily verified that the network is sign-skew-symmetric according to Eqs. (9) and (10). Thus, condition (i) in Definition 4 is satisfied. Second, for any two adjacent nodes, Eq. (10) can be rewritten as

$$w_j = w_i \left|\frac{a_{ji}}{a_{ij}}\right| := w_i r_{ij}. \tag{13}$$

For any $i \in \mathcal{V}$ and any path $(i_1, \dots, i_p, i)$, it follows from (13) that $w_i = r_{i_p i} w_{i_p}$, $w_{i_p} = r_{i_{p-1} i_p} w_{i_{p-1}}, \dots, w_{i_3} = r_{i_2 i_3} w_{i_2}$, $w_{i_2} = r_{i_1 i_2} w_{i_1}$. It follows that

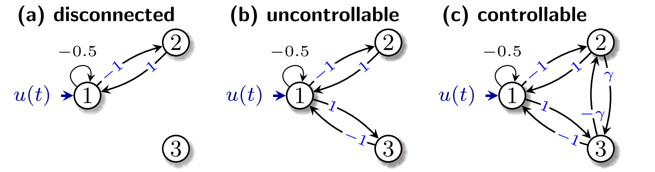
$$w_i = r_{i_1 i_2} r_{i_2 i_3} \dots r_{i_{p-1} i_p} r_{i_p i} w_{i_1} = r_{i_1 \dots i_p i} w_{i_1}. \tag{14}$$

If node i_1 is a control node, we have $w_{i_1} = 1/|2a_{i_1 i_1}|$ by (9). Substituting $w_{i_1} = 1/|2a_{i_1 i_1}|$ into (14) gives

$$w_i = r_{i_1 \dots i_p i}/|2a_{i_1 i_1}| = \beta_{i_1 \dots i_p i}, \tag{15}$$

where $\beta_{i_1 \dots i_p i}$ is the control impact along the path $(i_1, \dots, i_p, i)$ as in Definition 3. Equation (15) indicates that the control impacts of any control input $i_1 \in \mathcal{V}_c$ along any path $(i_1, \dots, i_p, i)$ are all equal to w_i. As a result, condition (ii) in Definition 4 is satisfied, and consequently the network is uniformly input-connected. In addition, since the value of $\beta_{i_1 \dots i_p i}$ depends merely on i, let $\beta_{i_1 \dots i_p i} := \beta_i$ and hence (15) implies (12).

(Sufficiency) Suppose a network is uniformly input-connected. We only need to show that the graphical conditions (i) and (ii) in Definition 4 imply equations (9) and (10). Let $D := \mathrm{diag}(\beta_1, \dots, \beta_n)$ where β_i is the control impact of node i. First, for any $i \in \mathcal{V}_c$, by considering the specific path (i, i), we have $\beta_i = 1/|2a_{ii}| \Rightarrow 2a_{ii}\beta_i = -1$. For any $i \in \mathcal{V} \setminus \mathcal{V}_c$, we have $a_{ii} = 0$ according to the definition of sign-skew-symmetric networks. Thus, equation (9) holds. Second, for any pair of adjacent nodes i and j, consider the path $(k, \dots, i, j)$ from an arbitrary control node k to node i and then to node j. The control impacts on nodes i and j satisfy $\beta_{k\dots ij} = |a_{ji}/a_{ij}|\beta_{k\dots i}$. Because $\beta_i = \beta_{k\dots i}$ and $\beta_j = \beta_{k\dots ij}$, we have $\beta_j = |a_{ji}/a_{ij}|\beta_i$ and consequently equation (10) holds. Since both (9) and (10) hold, the network is diagonally admissible and $D = \mathrm{diag}(\beta_1, \dots, \beta_n)$ is a solution of $AD + DA^{\mathrm{T}} = -BB^{\mathrm{T}}$. $\square$



(a) $A = \begin{bmatrix} -0.5 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & -0.5 & -0.75 \\ 0 & -1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} -0.5 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & -0.5 & -1.75 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} -0.5 & 1 & -1 \\ -1 & 0 & -\gamma \\ 1 & \gamma & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & -0.5 & -1.75 \\ 0 & -1 & 0.5-\gamma \\ 0 & 1 & -0.5-\gamma \end{bmatrix}$

Fig. 2. The three networks are all diagonally admissible for $W = \mathrm{diag}(1, 1, 1)$ because they satisfy $A + A^{\mathrm{T}} = -BB^{\mathrm{T}}$ where $B = [1, 0, 0]^{\mathrm{T}}$. Network (a) is disconnected and uncontrollable. Network (b) is connected but uncontrollable. Network (c) is controllable.

Theorem 3.1 establishes an equivalence between nodal energies and control impacts, and provides a graphical interpretation of the nodal energy in diagonally admissible networks. In particular, if a diagonally admissible network is stable, the matrix (12) is its controllability Gramian and, consequently, the ith nodal energy is

$$\varepsilon_i = \frac{1}{\beta_i}.$$

Therefore, a large control impact leads to a small nodal energy. Since the value of β_i is determined by the edge weights, we can obtain arbitrary nodal energies by selecting appropriate weights (see Section 5 for details).

Theorem 3.1 is only applicable to connected networks, where each input has control impacts on all the nodes. Yet, a diagonally admissible network may have disconnected components (see, for example, Fig. 2(a)). If each disconnected component has independent control inputs, then Theorem 3.1 can be applied independently to each of them (disconnected components with no control nodes are not of interest). In addition, Theorem 3.1 can be viewed as a generalization of the result in Kaszkurewicz and Hsu (1984), where the right-hand side of the Lyapunov equation $AW + WA^{\mathrm{T}} = -BB^T$ is assumed to be negative definite — in our setup, this condition corresponds to the case where there are n control nodes. Instead, Theorem 3.1 solves the case where the number of control nodes is less than n and, more importantly, establishes the equivalence between the network topology, its weights, and the energy needed to control the network to certain states.

4. Stability and controllability of diagonally admissible networks

In the previous section we present graphical conditions for diagonally admissible networks. Yet, there may exist an infinite number of diagonally admissible networks, with different network structures and edge weights, for the same diagonal Gramian. Further, it is not guaranteed that all diagonally admissible networks are stable and controllable (see Fig. 2 for an example). If a diagonally admissible network is controllable and stable, then the diagonal matrix (12) is its controllability Gramian; otherwise, the diagonal matrix (12) is still a solution to the Lyapunov equation although the controllability Gramian may not be well defined.

Motivated by the above discussion, in this section we study stability and controllability of diagonally admissible networks. Based on some basic properties of Lyapunov equations, two results about stability and controllability can immediately be obtained. First, the eigenvalues $\{\lambda_i\}_{i=1}^n$ of any $A \in \mathcal{D}(W, B)$ have non-positive real parts, that is, $\Re(\lambda_i) \leq 0$ for all i (Horn & Johnson, 1991, Lemma 2.4.5). As a result, a diagonally admissible network is either stable (i.e., $\Re(\lambda_i) < 0$ for all i) or marginally stable (i.e., $\Re(\lambda_i) = 0$ for some i). Second, for any diagonally admissible $A \in \mathcal{D}(W, B)$, we have that A is stable if and only if the pair (A, B) is controllable.[2] Given the equivalence between stability and controllability of diagonally admissible networks, we will next focus on controllability.

The next result shows that, for any W and B, the set $\mathcal{D}(W, B)$ is nonempty and it always contains controllable networks. We call a path that spans all the nodes in a network a *spanning path*, and refer to a sign-skew-symmetric network with one single spanning path as to a *chain network* (see Fig. 3 for an illustrative example).

Lemma 4.1 (*Existence of Controllable Networks in $\mathcal{D}(W, B)$*). *For any diagonal positive definite matrix W and input matrix B as in (2), the set $\mathcal{D}(W, B)$ always contains a controllable chain network.*

Proof. First, we construct a chain network that is diagonally admissible. Suppose $W = \mathrm{diag}(w_1, \ldots, w_n) > 0$. Assign the self-loop weights as $a_{ii} = -1/(2w_i) < 0$ for $i \in \mathcal{V}_c$, and $a_{ii} = 0$ otherwise. Select an arbitrary control node $i_1 \in \mathcal{V}_c$ and a spanning path $(i_1, \ldots, i_n)$ that starts from node i_1 (each node appears only once in the spanning path). For any adjacent nodes i and j in the path, select a_{ij} and a_{ji} such that $a_{ji}/a_{ij} = -w_j/w_i < 0$. The obtained network is a chain network and diagonally admissible for W because it satisfies (9) and (10). An illustrative example is given in Fig. 3.

Second, we prove the chain network constructed above is controllable. Although there may exist multiple inputs, we first examine the controllability of the chain network with the single input on node i_1. Due to the special structure of the chain network, by re-indexing the nodes appropriately, the adjacency matrix and input matrix can be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & a_{33} & a_{34} & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & a_{nn} \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

where $a_{ii} \neq 0$ if $i \in \mathcal{V}_c$ and $a_{ii} = 0$ otherwise. Since A is a tridiagonal matrix, the controllability matrix has the form of

$$K = [B_1, AB_1, A^2B_1, \ldots, A^{n-1}B_1]$$
$$= \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & a_{21} & * & \cdots & * \\ 0 & 0 & a_{21}a_{32} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \prod_{i=1}^{n-1} a_{(i+1)i} \end{bmatrix},$$

where $*$ denotes the entries that do not contribute to the rank of K. The matrix K is upper triangular and hence nonsingular for any nonzero edge weights. As a result, the chain network with the single input on node i_1 is controllable.

We next show that the network remains controllable when there are additional inputs. Consider the controllability Gramian $W_{t_f} = \int_0^{t_f} e^{At}BB^{\mathrm{T}}e^{A^{\mathrm{T}}t}\mathrm{d}t$. If $B = \mathrm{e}_{i_1}$, then $W_{t_f} = \int_0^{t_f} e^{At}e_{i_1}e_{i_1}^{\mathrm{T}}e^{A^{\mathrm{T}}t}\mathrm{d}t$, which is positive definite because the network is controllable in this case as shown above. If $B = [\mathrm{e}_{i_1}, \ldots, \mathrm{e}_{i_{n_c}}]$ where $i_1, \ldots, i_{n_c} \in \mathcal{V}_c$, then $W_{t_f} = \sum_{k=1}^{n_c}\int_0^{t_f} e^{At}e_{i_k}e_{i_k}^{\mathrm{T}}e^{A^{\mathrm{T}}t} \geq \int_0^{t_f} e^{At}e_{i_1}e_{i_1}^{\mathrm{T}}e^{A^{\mathrm{T}}t}\mathrm{d}t > 0$. Thus, the chain network with multiple control inputs is also controllable. $\square$

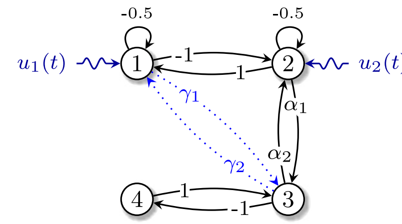


Fig. 3. Given $W = \mathrm{diag}(1, 1, 1, 1)$ and $B = [\mathrm{e}_1, \mathrm{e}_2]$, the chain network, which contains one single spanning path $(1, 2, 3, 4)$, is controllable and diagonally admissible for W when $\alpha_1/\alpha_2 = -1$. The network remains controllable and diagonally admissible for W when the dotted edges are added with $\gamma_1/\gamma_2 = -1$.

Based on Lemma 4.1, we are able to show that controllable and diagonally admissible networks form a dense set in $\mathcal{D}(W, B)$. Let $\|\cdot\|$ be an arbitrary matrix norm.

Theorem 4.2 (*Density of Controllable Networks in $\mathcal{D}(W, B)$*). *For any diagonal positive definite matrix W and input matrix B as in (2), the set of controllable networks in $\mathcal{D}(W, B)$ is dense. That is, for any $A_0 \in \mathcal{D}(W, B)$ and any $\epsilon > 0$, there exists $A_1 \in \mathcal{D}(W, B)$ such that (A_1, B) is controllable and $\|A_1 - A_0\| \leq \epsilon$.*

Proof. Suppose $A_2 \in \mathcal{D}(W, B)$ corresponds to a controllable network. Note A_2 always exists according to Lemma 4.1. Inspired by the proof in Lin (1974, Proposition 1), define

$$A_\alpha := (1-\alpha)A_0 + \alpha A_2, \quad \alpha \in [0, 1].$$

Note that $A_\alpha \in \mathcal{D}(W, B)$ for all $\alpha \in [0, 1]$ because $A_\alpha W + WA_\alpha^{\mathrm{T}} = -BB^{\mathrm{T}}$ for all α. For any $\epsilon > 0$, there exists $\alpha_0 \in [0, 1]$ such that $\|A_\alpha - A_0\| < \epsilon$ for all $\alpha \in [0, \alpha_0]$. Let $K(\alpha) = [B, A_\alpha B, \ldots, A_\alpha^{n-1}B]$ and $\phi(\alpha) = \det(K(\alpha)K^{\mathrm{T}}(\alpha))$. It is evident that $\phi(\alpha) = 0$ if and only if $K(\alpha)$ is row rank deficient, i.e., (A_α, B) is uncontrollable. Since the network is controllable when $\alpha = 1$, we have $\phi(1) \neq 0$ and hence $\phi(\alpha)$ is not identically zero. As a result, $\phi(\alpha) = 0$, which is a polynomial equation in one variable with finite order, has a finite number of zero roots. Consequently, there always exists $\alpha_1 \in [0, \alpha_0]$ such that $\phi(\alpha_1) \neq 0$, i.e., (A_{α_1}, B) is controllable. Thus, A_{α_1} corresponds to a controllable network and satisfies $\|A_{\alpha_1} - A_0\| \leq \epsilon$. $\square$

Theorem 4.2 lays a theoretical foundation for designing controllable diagonally admissible networks. In fact, given a diagonally admissible network, we can always obtain a controllable network by perturbing its adjacency matrix (the perturbation may or may not change the network structure). For example, the network in Fig. 2(b) is diagonally admissible but uncontrollable. We can perturb the weights between nodes 2 and 3 to obtain the controllable one in Fig. 2(c) (note that the perturbation of γ may be arbitrarily small). We remark that the controllability matrix of the perturbed network may be arbitrarily close to being rank deficient; yet, its nodal energies are guaranteed to equal the pre-specified values.

[2] Since any $A \in \mathcal{D}(W, B)$ satisfies $AW + WA^{\mathrm{T}} = -BB^{\mathrm{T}}$ and $W > 0$, this result follows from Theorem 2.4.7 and Remark 2.4.9 in Horn and Johnson (1991).

5. Design of stable and controllable networks with specified nodal energies

In this section we present a systematic way to construct diagonally admissible networks that are stable, thus controllable, with pre-specified nodal energies. Our network design problem is formally stated below.

Problem 1 (*Network Design Problem*)**.** Given a network with nodes $\mathcal{V}$, control nodes $\mathcal{V}_c \subseteq \mathcal{V}$, input matrix B as in (2), and desired nodal energies $\{\varepsilon_i\}_{i=1}^n$ with $\varepsilon_i > 0$, design the network adjacency matrix A such that A is stable and (A, B) is controllable with Gramian $W = \operatorname{diag}(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1})$. □

Before addressing Problem 1, we identify three edge operations that preserve diagonal admissibility. For any $A \in \mathcal{D}(W, B)$, it can be verified that $A + \Delta \in \mathcal{D}(W, B)$ if and only if

$$\Delta W + W\Delta^{\mathrm{T}} = 0. \tag{16}$$

Substituting $W = \operatorname{diag}(w_1, \ldots, w_n)$ into (16) gives

$$[\Delta]_{ij} w_j = -[\Delta]_{ji} w_i. \tag{17}$$

Eq. (17) leads to the following edge operations:

(o1) *Edge scaling:* For any pair of adjacent nodes i and j, change the edge weights a_{ij} and a_{ji} to αa_{ij} and αa_{ji},with $\alpha \in \mathbb{R}_{\neq 0}$. In this case, $[\Delta]_{ij} = (\alpha - 1)a_{ij}$ and $[\Delta]_{ji} = (\alpha - 1)a_{ji}$.

(o2) *Edge addition:* For any pair of nodes i and j that are not adjacent (i.e., $a_{ij} = a_{ji} = 0$), add two edges with opposite directions between them and set the edge weights to $a_{ij} = \alpha$ and $a_{ji} = -\alpha(w_j/w_i)$, with $\alpha \in \mathbb{R}_{\neq 0}$. In this case, $[\Delta]_{ij} = \alpha$ and $[\Delta]_{ji} = -\alpha(w_j/w_i)$.

(o3) *Edge removal:* For any pair of adjacent nodes i and j, delete the existing edges between them, i.e., set $a_{ij} = a_{ji} = 0$. In this case, $[\Delta]_{ij} = -a_{ij}$ and $[\Delta]_{ji} = -a_{ji}$.

It can be easily shown that, if a network belongs to $\mathcal{D}(W, B)$, then it still belongs to $\mathcal{D}(W, B)$ after performing any of the above edge operations. These operations will be later used in our design algorithm.

The following result shows that edge operations (o1) and (o2) generically preserve network controllability. Operation (o3), instead, may render the network disconnected and hence prevent controllability.

Proposition 5.1 (*Edge Operations that Preserve Controllability*)**.** *Let $A \in \mathcal{D}(W, B)$ for some diagonal positive definite matrix W and input matrix B as in (2). Let (A, B) be controllable and let $A_1 = A + \Delta$, where Δ is obtained through the operations (o1)–(o2). Then, $A_1 \in \mathcal{D}(W, B)$ and (A_1, B) is controllable for all $\alpha \in \mathbb{R}_{\neq 0}$ except at most a finite number of values of α.*

Proof. In the case of operation (o1), the weights a_{ij}, a_{ji} are changed to $\alpha a_{ij}, \alpha a_{ji}$ with $\alpha \in \mathbb{R}_{\neq 0}$. Let $K(\alpha) = [B, A_1 B, \ldots, A_1^{n-1}B]$ and $\phi(\alpha) = \det(K(\alpha)K^{\mathrm{T}}(\alpha))$. It is evident that $\phi(\alpha) = 0$ if and only if $K(\alpha)$ is row rank deficient, i.e., (A_1, B) is uncontrollable. Since the network is controllable when $\alpha = 1$, we have $\phi(1) \neq 0$ and hence $\phi(\alpha)$ is not identically zero. Because $\phi(\alpha) = 0$ is a polynomial equation in one variable with finite order, there only exist a finite number of α where $\phi(\alpha) = 0$ (i.e., (A_1, B) is uncontrollable). The case of operation (o2) can be proved analogously. □

Since edge operations (o1) and (o2) generically preserve network controllability, they provide degrees of freedom to design network structures and edge weights of diagonally admissible networks. To illustrate Proposition 5.1, consider the network in Fig. 3. For nodes 2 and 3, as long as $\alpha_1/\alpha_2 = -1$, the absolute values of α_1 and α_2 can arbitrarily be changed while the network remains controllable with Gramian $W = I_4$. Further, adding two edges with weights satisfying $\gamma_1/\gamma_2 = -1$ between nodes 1 and 3 does not alter the Gramian of the network either.

Algorithm 1 Design of stable and controllable networks with specified diagonal Gramians

Require: The specified diagonal Gramian is $W = \operatorname{diag}(w_1, \ldots, w_n) > 0$.

1: **Design self-loops:** Let $a_{ii} = -1/(2w_i) < 0$ for control node $i \in \mathcal{V}_c$, and $a_{ii} = 0$ for other nodes.

2: **Design a spanning path:** Select a control node $i_1 \in \mathcal{V}_c$ and design a path $(i_1, \ldots, i_n)$ that spans all the nodes in the network and where each node appears only once.

3: **Design edge weights for the path:** Suppose i and j are two adjacent nodes in the path. Select a_{ij} and a_{ji} such that $a_{ji}/a_{ij} = -w_j/w_i < 0$. There are no constraints on the absolute values of a_{ij} or a_{ji}.

4: **Add extra edges:** If desired, extra edges can be added. To add edges between nodes i and j, the edge weights must satisfy $a_{ji}/a_{ij} = -w_j/w_i < 0$.

We propose Algorithm 1 to design networks that are stable, controllable, and diagonally admissible for arbitrary diagonal positive definite matrices. Algorithm 1 consists of four steps. In the first three steps, a chain network that contains a single spanning path is constructed. This chain network is controllable and features the specified diagonal Gramian according to Theorem 3.1 and Lemma 4.1.[3] The fourth step adds extra edges to the chain network to create various network structures. Adding these edges preserves the controllability Gramian according to Proposition 5.1.

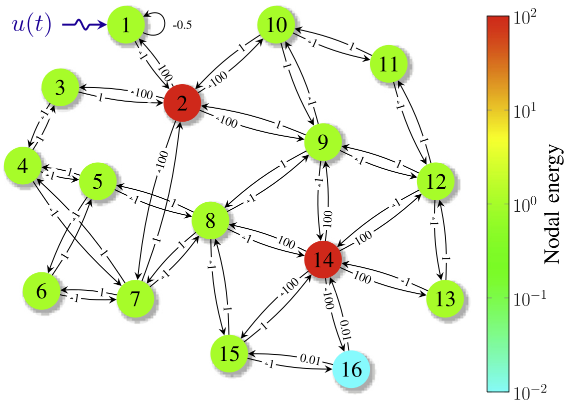
An illustrative example of Algorithm 1 is in Fig. 4. In this example, the desired nodal energies are specified as $\epsilon_1 = \epsilon_{14} = 100$, $\epsilon_{16} = 0.001$, and $\epsilon = 1$ for the remaining nodes. Following the first three steps of Algorithm 1, the spanning path $(1, 2, 3, \ldots, 16)$ is first constructed, in a way that the nodal energy of this spanning path is as desired. Then, following the fourth step of Algorithm 1, extra edges are added to finally generate the network as shown in Fig. 4(a).

The networks constructed by Algorithm 1 are not unique. Given a set of desired nodal energies, there exist an infinite number of controllable and stable networks featuring these nodal energies. In fact, adding extra edges or carefully scaling the weights of existing edges may not change nodal energies. This provides freedom to design networks satisfying other constraints or performance objectives. For example, weights can be designed to lie within certain intervals imposed by physical constraints, or optimized to achieve certain performance objectives.

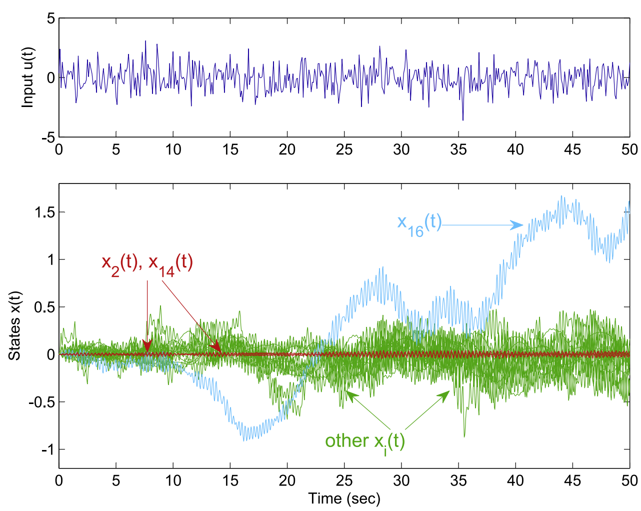
Algorithm 1 requires one control node to achieve arbitrarily specified nodal energies. This is because the first three steps in Algorithm 1 construct a chain network that is controllable with one single control node. Although multiple inputs are redundant to achieve specified nodal energies, they may affect other properties such as the spectrum and the robustness of the adjacency matrix (adding an input node would also add a negative diagonal entry to the adjacency matrix). Finally, Algorithm 1 requires only simple operations, and it can be used efficiently to construct large networks.

Remark 1 (*Relevance of the Results in the Broader Context of Network Analysis and Control*)**.** The results presented in this paper, particularly the graphical conditions and the network design algorithm,

[3] In Algorithm 1, the spanning path can be replaced by any controllable structure with weights satisfying the conditions in Theorem 3.1 for W.



(a) Color-coded nodal energies: $\varepsilon_2 = \varepsilon_{14} = 100$, $\varepsilon_{16} = 0.01$, and $\varepsilon_i = 1$ for other i. The path $(1, 2, 3, \ldots, 16)$ spans all the nodes.



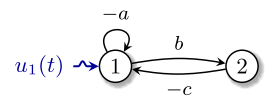
(b) State response to white-noise input

Fig. 4. Design of a stable and controllable network with specified nodal energies by Algorithm 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

reveal important controllability properties and tradeoffs of continuous time linear networks. For instance, our results rigorously lead to the following novel observations:

(i) *There exist a class of continuous-time networks whose control energy is independent of their cardinality and number of control nodes.* In particular, following Algorithm 1, we can construct arbitrarily large networks with a single control input and arbitrary nodal energies. This is a counterexample to the conclusion drawn in Yan et al. (2015), among other papers, that the control energy of complex networks increases rapidly with the network cardinality. Further, this finding constitutes the continuous-time counterpart of the result presented in Pasqualetti and Zampieri (2014) for certain anisotropic discrete-time networks.

(ii) *The nodal energy of a node can be independent of its graphical distance from the control nodes.* As illustrated in Fig. 4(a), nodes that are located close to the control node (for example, node 2) may have high nodal energy while nodes that are far away from the control node (for example, node 12) may have low nodal energy. This property provides a note of caution, for instance, for the procedure proposed in Chen, Wang, Wang, and Lai (2016) to reduce the energy to control a network.



$$A = \begin{bmatrix} -a & -c \\ b & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; W = \frac{1}{2a}\begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{c} \end{bmatrix}, \; K = \begin{bmatrix} 1 & -a \\ 0 & b \end{bmatrix}$$

Fig. 5. The controllability matrix K approaches to rank deficient while the controllability Gramian W remains well conditioned when $b, c \to 0$ and b/c is unchanged $(a, b, c > 0)$.

(iii) *The controllability Gramian and the controllability matrix can provide different or even opposite measures of the controllability degree of a network.* In particular, as shown in Fig. 5, the controllability matrix may become arbitrarily close to being rank deficient, suggesting that the network approaches uncontrollability, whereas the controllability Gramian remains well conditioned, indicating that the network requires control inputs with small energy. This property becomes particularly important when assessing controllability numerically, and it further motivates the development of analytical, thus numerically reliable, controllability tests for networks.

(iv) *Networks with one control node and diagonal controllability Gramian typically have small stability margins.* For example, although the network in Fig. 4(a) is stable, its rightmost eigenvalue is very close to the imaginary axis (its real part is 0.0015), indicating that a small perturbation of the network structure or edge weights could render the network unstable. This observation is in line with the results of Pasqualetti, Favaretto, Zhao, and Zampieri (2018), where it is shown that there exists a fundamental tradeoff between the controllability degree of a network and its robustness to parameter changes.

Finally, our work addresses, for the first time, the problem of designing sparse systems (networks) with prescribed, quantitative controllability properties (Gramian), and it shows how a solution to this problem can be used to design selectively secure networks against disturbances (for example, in Fig. 4(a), nodes 2 and 14 are purposively protected against the effect of a white noise input entering node 1). While the design of non-sparse dynamical systems with diagonal Gramian has been previously studied, sparsity constraints in the system matrices are considerably more difficult to account for and have not been sufficiently investigated prior to this work. □

6. Physical networks with diagonal Gramian

In this section, we present synthetic and physical networks featuring a diagonal controllability Gramian.

6.1. Natural networks with diagonal Gramian

In many cyber–physical networks, such as wireless sensor networks or multi-robot systems, the edge weights are usually stored as parameters in software programs and can be set easily to achieve any desired pattern of nodal energies. As a matter of fact, a sign-skew-symmetric interaction pattern exists in many natural networks. For example, in ecological food webs, predator–prey interactions have the sign-skew-symmetric structure (Allesina et al., 2008; Kaszkurewicz & Bhaya, 2000). Specifically, for a pair of prey and predator species, the population of the preys has positive

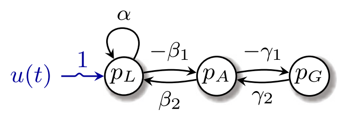


Fig. 6. A simplified model of a predator–prey food chain.

impact on that of the predators because more preys provide more food for the predators. On the other hand, the population of the predators has negative impact on that of the preys because more predators would consume more preys.

Consider a food chain of three species: lion, antelope, and grass. Let p_L, p_A, and p_G be the populations of the three species, respectively. Due to the sign-skew-symmetric interactions between predictor and pray populations, the dynamics of the food chain can be described by the following linear system,

$$\begin{bmatrix} \dot{p}_L \\ \dot{p}_A \\ \dot{p}_G \end{bmatrix} \begin{bmatrix} -\alpha & \beta_2 & 0 \\ -\beta_1 & 0 & \gamma_2 \\ 0 & -\gamma_1 & 0 \end{bmatrix} \begin{bmatrix} p_L \\ p_A \\ p_G \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u,$$

where $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2$ are all positive. The network is depicted in Fig. 6. This model is simplified because it ignores the self-regulation of the populations of antelopes and grass. For this simplified model, it follows from Theorem 3.1 that the Gramian is diagonal and has the form

$$W = \frac{1}{2\alpha}\mathrm{diag}\left(1, \quad \frac{\beta_1}{\beta_2}, \quad \frac{\beta_1\gamma_1}{\beta_2\gamma_2}\right).$$

As a result, the energies to control the populations of the lions, antelopes, grass are, respectively, 2α, $2\alpha\beta_2/\beta_1$, $2\alpha\beta_2\gamma_2/(\beta_1\gamma_1)$.

6.2. Technological networks with diagonal Gramian

Sign-skew-symmetric structures also exist in circuits and mechanical networks (Kaszkurewicz & Bhaya, 2000). We next give two examples and derive their diagonal Gramians. Fig. 7(a) shows a resistor–inductor–capacitor (RLC) circuit. The state of the circuit consists of $\upsilon_i(t)$ and $\iota_i(t)$, which are the voltage of capacitor i and current of inductor i, respectively. The input $u(t)$ is the voltage applied to the resistor. The notations R, L_i, and C_i denote the resistance, inductance, and capacitance, respectively. A graphical representation of the circuit is shown in Fig. 7(b). The dynamics of the circuit can be written as

$$\begin{bmatrix} \dot{\iota}_1 \\ \dot{\upsilon}_1 \\ \dot{\iota}_2 \\ \dot{\upsilon}_2 \\ \vdots \\ \dot{\iota}_n \\ \dot{\upsilon}_n \end{bmatrix} = \begin{bmatrix} \frac{R}{-L_1} & \frac{1}{-L_1} & & & & & \\ \frac{1}{C_1} & 0 & \frac{1}{-C_1} & & & & \\ & \frac{1}{L_2} & 0 & \frac{1}{-L_2} & & & \\ & & \frac{1}{C_2} & 0 & \frac{1}{-C_2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{1}{L_n} & 0 & \frac{1}{-L_n} \\ & & & & & \frac{1}{C_n} & 0 \end{bmatrix} \begin{bmatrix} \iota_1 \\ \upsilon_1 \\ \iota_2 \\ \upsilon_2 \\ \vdots \\ \iota_n \\ \upsilon_n \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u. \quad (18)$$

By applying Theorem 3.1, we find that the Gramian is diagonal and satisfies

$$W = \frac{1}{2R}\mathrm{diag}\left(\frac{1}{L_1}, \frac{1}{C_1}, \frac{1}{L_2}, \frac{1}{C_2}, \dots, \frac{1}{L_n}, \frac{1}{C_n}\right).$$

As a result, the nodal energies required to control the current ι_i and the voltage υ_i can be, respectively, expressed in simple forms as $\varepsilon_{\iota_i} = 2RL_i$ and $\varepsilon_{\upsilon_i} = 2RC_i$. It should be noted that the weight for each control input is assumed to be 1 in Theorem 3.1, whereas the control input weight in the RLC network is $1/L_1$. In order to handle the weighted input, we may simply treat $u(t)/L_1$ as a new input, so that the energy obtained from Theorem 3.1 is $\int_0^\infty u^2(t)/L_1^2 \mathrm{d}t$.

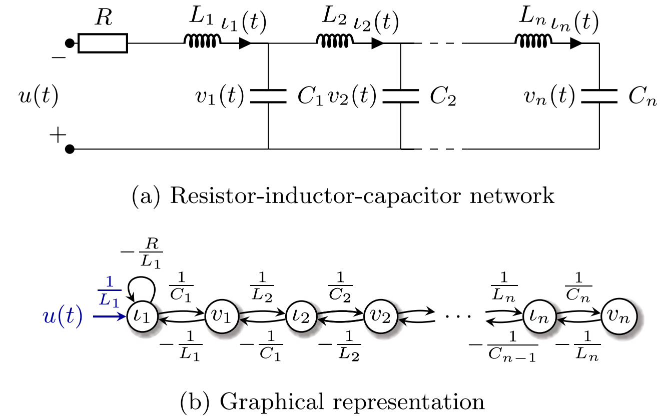


Fig. 7. A RLC circuit which is stable, controllable, and diagonally admissible.

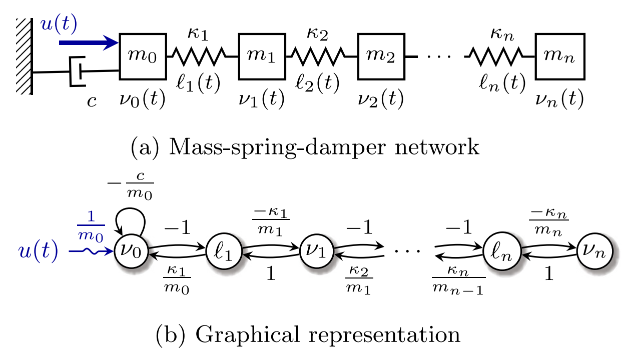


Fig. 8. A mass–spring–damper mechanical network which is stable, controllable, and diagonally admissible.

Fig. 8(a) shows a mass–spring–damper mechanical network. The states of the network consist of $\nu_i(t)$ and $\ell_i(t)$, which are the velocity of mass i and deformation of spring i, respectively. The input $u(t)$ is the force applied to the first mass. The notations c, m_i, and κ_i denote the damping ratio, mass, and spring stiffness, respectively. A graphical representation of the network is given in Fig. 8(b). By Theorem 3.1, the Gramian is diagonal and equal to

$$W = \frac{1}{2c}\mathrm{diag}\left(\frac{1}{m_0}, \frac{1}{\kappa_1}, \frac{1}{m_1}, \dots, \frac{1}{\kappa_n}, \frac{1}{m_n}\right).$$

The nodal energies required to control the velocity ν_i of the mass m_i and the deformation ℓ_i of the spring i are $\varepsilon_{\nu_i} = 2cm_i$ and $\varepsilon_{\ell_i} = 2c\kappa_i$, respectively.

It is worth emphasizing two interesting features of the RLC and mass–spring networks. First, either of the networks has a chain structure and a single control input. Such kind of chain networks always have diagonal Gramians for arbitrary edge weights because there is only one single path from the control input to each node. Second, either of the networks has one single energy-consuming component (i.e., the resistor and damper), while all the other components do not consume any energy. The energy-consuming components correspond to the negative self-loops of the control nodes, whose role is to ensure system stability. Without these components, the energy in the networks does not vanish and the networks continuously oscillate.

7. Conclusions

In this paper we characterize novel relations between the graphical structure of a network and its controllability Gramian, and we propose an algorithm to design networks with pre-specified controllability properties. In particular, the main technical contribution of this paper is to derive necessary and sufficient graphical conditions for a network to be controllable with a diagonal and prespecified controllability Gramian. We derive precise expression of the nodal energies, that is, the control energy needed to change the state of a single node, in terms of edge weights along the paths starting from the control nodes. We also propose an algorithm to construct stable and controllable networks with desired nodal energies. Because nodal energies quantify the $\mathcal{H}_2$ norm of the responses of individual nodes to white-noise or impulsive inputs, our network design algorithm can be used in multi-agent applications to selectively enforce robustness and security of the nodes against disturbances.

In a broader context, our study reveals novel controllability properties and tradeoffs of complex networks. For example, contrary to previous studies, our results show that there exists a class of continuous-time networks whose control energy is independent of the size of the network and number of control nodes, and that the energy to control a node can be made independent of its distance from the control nodes. Several research directions are envisioned, including the generalization of our analysis to more general sparse Gramian structures, and the design of distributed algorithms to modify the network weights to dynamically change nodal energies.

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