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On the Number of Strongly Structurally Controllable Networks

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Abstract— **Network controllability is a structural property, that is, mild and well-understood conditions on the network interconnection pattern ensure controllability from a given set of control nodes for *most* choices of the edge weights. To ensure network controllability for *all* choices of edge weights, namely strong structural controllability, more stringent connectivity conditions need to be satisfied. In this paper we derive an alternative algebraic characterization of strong structural controllability of networks with self-loops. This characterization allows us to systematically enumerate all strongly structurally controllable networks with given cardinality and number of control nodes. Differently from the case of (weak) structural controllability we show that, when the ratio of control nodes to the total number of nodes converges to zero, then the fraction of strongly structurally controllable networks decreases to zero. Conversely, when the ratio of control nodes to the total number of nodes converges to one, then the fraction of strongly structurally controllable networks remains lower bounded. Altogether, the results in this paper complement existing studies on the asymptotic number of controllable graphs.**

I. Introduction

Network systems are commonly used to model complex natural and technological systems. Examples range from biological networks [1], brain networks [2], the smart grid [3], to social networks [4]. Due to dimensionality, unknown dynamics, and uncertain parameters, control of complex networks is a challenging problem that is attracting considerable attention from different research communities. Here control refers to the possibility of injecting targeted control inputs through few nodes, so as to manipulate the entire network configuration and ensure reliability and performance.

Controllability of networks depends on the network structure as well as on the dynamics of the network nodes. To highlight the fundamental role of the network structure, in this paper we focus on networks where the dynamics are linear and specified by a weighted adjacency matrix of the underlying graph. In this setting, network controllability becomes a structural property [5] that can be tested based on the interconnection structure and independently of the edge weights. In fact, from the well-established theory of structural control [6], we know that mild connectivity conditions on the interconnection pattern guarantee controllability for almost all choices of the edge weights, provided that the weights can be selected independently from each other. When edge weights are mutually dependent or constrained, the classic structural theory no longer applies, and a stronger notion of controllability, namely strong structural controllability, is needed to guarantee network controllability from a set of control nodes and for all choices of the network weights. In this paper, we focus on this stronger notion, providing a systematic procedure to enumerate all strongly structurally controllable networks with self-loops.

Related work The topic of network controllability has sparked interest across different research communities. The literature can be classified into *qualitative* and *quantitative* studies. Qualitative approaches adopt the binary controllability notion, first introduced in [7], and employ graphical and combinatorial techniques, e.g., see [8], [9], [10], [11]. On the other hand, quantitative studies use a graded metric of controllability and typically leverage control-theoretic methods, e.g., see [12], [13], [14], [15]. This work falls within the first class, understanding algebraic and graph-theoretic properties of strong structural controllability.

The notion of strong structural controllability is used to ensure that all networks with given interconnections structure are controllable from a set of control nodes, independently of the edge weights [16]. Early results on strong structural controllability are presented in [17], [18], where graphical conditions to ensure such property are provided. In [19] existing results on strong structural controllability are found incorrect, and a different characterization using the notion of *cycle families* is given. One practical extension of previous work to the multi-input case is provided in [20], together with algebraic conditions, also given in [21]. Furthermore, [22] and [23] present conditions and algorithms with polinomial complexity based on the notions of *constrained matching* and *zero-forcing set*, respectively, and a further step is attained in [24], where linear complexity is achieved. Finally, strong structural controllability is presented for time-varying systems in [25]. With respect to the existing work, we provide a refined condition for strong structural controllability of networks with self loops. This characterization is later used to characterize the asymptotic number of strongly structurally controllable networks.

Another area of research related to this work consists of the enumeration of graphs satisfying certain properties. For instance, [26], [27] show that for the weaker controllability notion the relative number of controllable graphs compared to the total number of simple graphs on n nodes approaches one, as n tends to infinity. We complement this line of research by demonstrating that this is not the case for strong structural controllability.

Contribution The contribution of this paper is twofold. First, we provide an algebraic characterization of strong structural controllability for networks with self-loops. This characteri-

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zation leads to an efficient algorithm to test for strong structural controllability of a given network. Second, we exploit our result to enumerate all strongly structurally controllable networks with given cardinality and number of control nodes. We show that the ratio of strongly structurally controllable networks to the total number of networks behaves as the ratio of number of control nodes m to the total number of nodes n. In particular, when m/n converges to zero, so does the ratio of strongly structurally controllable networks to the total number of networks. Conversely, when m/n converges to one, the ratio of strongly structurally controllable networks to the total number of networks remains bounded from below.

Paper organization The rest of the paper is organized as follows. Section II describes our setup and introduces some preliminary notions. Section III contains our algebraic characterization of strong structural controllability. In Section IV we enumerate all strongly structurally controllable networks with given cardinality and number of control nodes. Finally, Section V concludes the paper.

II. Problem setup and preliminary notions

We model a network with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertices and edges sets, respectively. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be the weighted adjacency matrix of $\mathcal{G}$, with $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, and $a_{ij} \in \mathbb{R}$ otherwise. We let $\mathcal{K} = \{k_1, \dots, k_m\} \subseteq \{1, \dots, n\}$ denote the set of control nodes, and let the system dynamics evolve according to the linear model

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $B \in \mathbb{R}^{n \times m}$ is the input matrix defined as $B = \begin{bmatrix} e_{k_1} & \dots & e_{k_m} \end{bmatrix}$, and e_i is the i-th canonical vector.

We define the n-steps controllability matrix

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, \tag{2}$$

and we recall that the pair (A, B) is controllable if and only if $\mathcal{C}(A, B)$ is full rank [28].

We will be making use of the theory of structure matrices and their generic properties. We denote with bold symbols $\mathbf{M} = [\mathbf{m}_{ij}]$ a *structure matrix* [6], [16]: its entries are either zero or indeterminate (nonzero) values. The latter are here denoted by the symbol $\times$. Finally, the real matrix $M = [m_{ij}]$ is an admissible numerical realization of $\mathbf{M}$, in other words $M \in \mathbf{M}$, if it can be obtained by assigning some *nonzero* values to the indeterminate entries of $\mathbf{M}$.

The following concepts will be used throughout the paper. First, the structure pair $(\mathbf{A}, \mathbf{B})$ is *strongly structurally controllable* (SSC) if all admissible numerical realizations $(A, B) \in (\mathbf{A}, \mathbf{B})$ are controllable. Second, we will say that the pair (A, B) is *permutation-similar* to $(\tilde{A}, \tilde{B})$ if there exists a permutation matrix P satisfying $\tilde{A} = PAP^T$ and $\tilde{B} = PB$. Also, with a slight abuse of notation, we will say that the structure pair $(\mathbf{A}, \mathbf{B})$ is permutation-similar to $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ if every admissible realization of $(\mathbf{A}, \mathbf{B})$ is permutation-similar to some admissible realization of $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$. Third, the structure pair $(\mathbf{A}, \mathbf{B})$ uniquely identifies an unweighted labeled graph with m control nodes. Finally, we will make use of the following notation: $\mathbf{A}(i : \text{end}, j) \in \mathbb{R}^{n-i+1}$ denotes the column vector with entries $\mathbf{a}_{kj}$, where $k = i, \dots, n$.

III. Algebraic characterization of strong structural controllability

In this section, we provide an algebraic characterization of strongly structurally controllable networks with self loops. It should be observed that, although conditions for more general networks exist, e.g., see [21], [23], [29], [30], our characterization is in fact instrumental for the main results in Section IV. We restrict our analysis to input-connectable networks [6] with self-loops and independent control nodes. Formally, we make the following assumptions:

(A1) the diagonal entries of $\mathbf{A}$ are nonzero;

(A2) the structure input matrix is $\mathbf{B} = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_m \end{bmatrix}$, where $\mathbf{e}_i$ is the structure vector associated with e_i;

(A3) every admissible realization of $(\mathbf{A}, \mathbf{B})$ is input-connectable, that is, there exists a path[1] from a control node $i \in \{1, \dots, m\}$ to every node $j \in \{1, \dots, n\}$ in the unweighted graph associated with such pair.

Assumption (A1) allows to address the problem from an algebraic point of view, and yields conditions that are equivalent to those pointed out in [18], [19], [21], [22]. Assumptions (A2) and (A3) are not restrictive in general, in fact, a nodes relabeling suffices for (A2) to hold, while Assumption (A3) is necessary for (structural) controllability [6], thus for the stronger notion.

The following theorem provides an algebraic condition equivalent to strong structural controllability of $(\mathbf{A}, \mathbf{B})$.

Theorem 3.1: ***(Strong structural controllability)*** Let the structure pair $(\mathbf{A}, \mathbf{B})$ satisfy Assumptions (A1)-(A3). Then, the following statements are equivalent:

(i) $(\mathbf{A}, \mathbf{B})$ is strongly structurally controllable;

(ii) $(\mathbf{A}, \mathbf{B})$ is permutation-similar to $(\tilde{\mathbf{A}}, \mathbf{B})$, and for all $i \in \{m+1, \dots, n\}$ there exists $j \in \{1, \dots, i-1\}$ such that[2]

$$\text{supp}(\tilde{\mathbf{A}}(i : \text{end}, j)) = \{1\}.$$

The proof of Theorem 3.1 is postponed to Appendix.

It is worth noting that, when the network is controlled from a single node, Theorem 3.1 implies that $(\mathbf{A}, \mathbf{B})$ is strongly structurally controllable if and only if $(\mathbf{A}, \mathbf{B})$ is permutation-similar to $(\tilde{\mathbf{A}}, \mathbf{B})$ where $\tilde{\mathbf{A}}$ is upper Hessenberg [17].

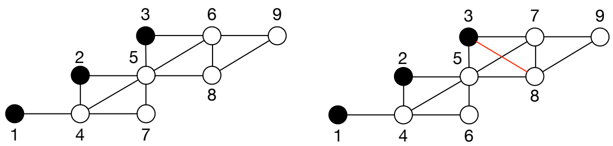
We now illustrate Theorem 3.1 through two examples.

Example 1: ***(SSC network)*** Consider the network shown in Fig. 1(a) with control set $\mathcal{K} = \{1, 2, 3\}$ and structure adjacency matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & 0 & 0 & \mathbf{a}_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}_{22} & 0 & \mathbf{a}_{24} & \mathbf{a}_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}_{33} & 0 & \mathbf{a}_{35} & \mathbf{a}_{36} & 0 & 0 & 0 \\ \mathbf{a}_{41} & \mathbf{a}_{42} & 0 & \mathbf{a}_{44} & \mathbf{a}_{45} & 0 & \mathbf{a}_{47} & 0 & 0 \\ 0 & \mathbf{a}_{52} & \mathbf{a}_{53} & \mathbf{a}_{54} & \mathbf{a}_{55} & \mathbf{a}_{56} & \mathbf{a}_{57} & \mathbf{a}_{58} & 0 \\ 0 & 0 & \mathbf{a}_{63} & 0 & \mathbf{a}_{65} & \mathbf{a}_{66} & 0 & \mathbf{a}_{68} & \mathbf{a}_{69} \\ 0 & 0 & 0 & \mathbf{a}_{74} & \mathbf{a}_{75} & 0 & \mathbf{a}_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}_{85} & \mathbf{a}_{86} & 0 & \mathbf{a}_{88} & \mathbf{a}_{89} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{a}_{96} & 0 & \mathbf{a}_{98} & \mathbf{a}_{99} \end{bmatrix}.$$

[1]A path is a sequence of vertices $\{v_1, v_2, \dots, v_p\}$ such that (v_i, v_{i+1}) belongs to the edge set, for all $i \in \{1, \dots, p-1\}$.

[2]The support of the vector $x \in \mathbb{R}^n$, namely $\text{supp}(x)$, is the set of indices $i \in \{1, \dots, n\}$ such that $x_i \neq 0$.



(a) (b)

Fig. 1. Control nodes are in black. (a) The network is SSC. (b) The network is not SSC because of the edge between nodes 3 and 8. See Example 1 and 2.

Condition (ii) in Theorem 3.1 concludes that the pair $(\mathbf{A},\mathbf{B})$ is strongly structurally controllable. Indeed, for all $i \in \{4,\dots,9\}$, $\text{supp}(\mathbf{A}(i:\text{end}, i-3)) = \{1\}$, as highlighted by grey columns. Moreover, the controllability matrix reads

$$\mathcal{C}(\mathbf{A},\mathbf{B}) = \underbrace{\begin{bmatrix} \times & \otimes & \otimes & \cdots & \otimes & \otimes & \cdots & \otimes \\ 0 & \times & \otimes & \cdots & \otimes & \otimes & \cdots & \otimes \\ 0 & 0 & \times & \cdots & \otimes & \otimes & \cdots & \otimes \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \times & \otimes & \cdots & \otimes \end{bmatrix}}_{\mathbf{C}^*},$$

where $\otimes$ denotes an entry that can be either zero or nonzero. Notice that diagonal elements of $\mathbf{C}^* = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix}$ are nonzero for every admissible numerical realization and equal to $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_1\mathbf{a}_{41}$, $\mathbf{b}_2\mathbf{a}_{52}$, $\mathbf{b}_3\mathbf{a}_{63}$, $\mathbf{b}_1\mathbf{a}_{41}\mathbf{a}_{74}$, $\mathbf{b}_2\mathbf{a}_{52}\mathbf{a}_{85}$, $\mathbf{b}_3\mathbf{a}_{63}\mathbf{a}_{96}$, respectively.

Since $\mathcal{C}(A,B)$ is full rank for every admissible realization $(A,B) \in (\mathbf{A},\mathbf{B})$, the claimed statement follows. □

The following example illustrates the necessity of statement (ii) to ensure controllability. Indeed, notice that if there exists some index $i \in \{m+1,\dots,n\}$ such that $|\text{supp}(\mathbf{A}(i:\text{end}, j))| \neq 1$ holds for all $j \in \{1,\dots,i-1\}$, then the pair $(\mathbf{A},\mathbf{B})$ is not strongly structurally controllable.

*Example 2: **(Non-SSC network)*** Consider the network in Fig. 1(b) with control nodes $\mathcal{K} = \{1,2,3\}$ and structure adjacency matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & 0 & 0 & \mathbf{a}_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}_{22} & 0 & \mathbf{a}_{24} & \mathbf{a}_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}_{33} & 0 & \mathbf{a}_{35} & 0 & \mathbf{a}_{37} & \mathbf{a}_{38} & 0 \\ \mathbf{a}_{41} & \mathbf{a}_{42} & 0 & \mathbf{a}_{44} & \mathbf{a}_{45} & \mathbf{a}_{46} & 0 & 0 & 0 \\ 0 & \mathbf{a}_{52} & \mathbf{a}_{53} & \mathbf{a}_{54} & \mathbf{a}_{55} & \mathbf{a}_{56} & \mathbf{a}_{57} & \mathbf{a}_{58} & 0 \\ 0 & 0 & 0 & \mathbf{a}_{64} & \mathbf{a}_{65} & \mathbf{a}_{66} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}_{73} & 0 & \mathbf{a}_{75} & 0 & \mathbf{a}_{77} & \mathbf{a}_{78} & \mathbf{a}_{79} \\ 0 & 0 & \mathbf{a}_{83} & 0 & \mathbf{a}_{85} & 0 & \mathbf{a}_{87} & \mathbf{a}_{88} & \mathbf{a}_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a}_{97} & \mathbf{a}_{98} & \mathbf{a}_{99} \end{bmatrix}.$$

Notice that $\text{supp}(\mathbf{A}(i:\text{end}, j)) \neq \{1\}$ for $i = 7$ and $j \in \{1,\dots,6\}$. It follows that the network is not strongly structurally controllable, as highlighted by the grey columns. For instance, it can be verified that every realization with $a_{ij} = 1$ for all $i,j \in \{7,8,9\}$, $a_{73} = a_{75} = 1$, and $a_{83} = a_{85} = -1$ is not controllable. □

The necessity of statement (ii) can be further employed to design a procedure to test whether a pair $(\mathbf{A},\mathbf{B})$ is strongly structurally controllable, as we do in Algorithm 1.

Algorithm 1: *Test for strong structural controllability*

Input : Structure pair $(\mathbf{A},\mathbf{B})$;
Require : The pair $(\mathbf{A},\mathbf{B})$ satisfies Assumptions (A1)-(A3);
Output : Whether $(\mathbf{A},\mathbf{B})$ is strongly structurally controllable;

for $i = m+1 : n$ **do**
  if $|\textit{supp}(A(i:\textit{end}, j))| \neq 1$ *for all* $j \in \{1,\dots,i-1\}$ **then**
    return $(\mathbf{A},\mathbf{B})$ is not strongly structurally controllable;
  else
    relabel the nodes such that $\text{supp}(A(i:\text{end}, j)) = \{1\}$ for some $j \in \{1,\dots,i-1\}$;
return $(\mathbf{A},\mathbf{B})$ is strongly structurally controllable;

IV. ENUMERATING STRONGLY STRUCTURALLY CONTROLLABLE NETWORKS

In this section, we provide lower and upper bounds on the number of distinct networks that are strongly structurally controllable with n nodes and m control nodes. Further, we show that the fraction of strongly structurally controllable networks converges to zero whenever $\lim_{n\to\infty} m/n < 1$.

Theorem 3.1 implies that the adjacency matrix of a strongly structurally controllable network has a very specific structure. In fact, at least $n-m$ columns of the adjacency matrix have zero entries beyond a certain row index, and are located within a certain range of columns locations.

*Lemma 4.1: **(Fixed structure columns)*** Let $|\mathcal{K}| = m$. Then, there exist at least m^{n-m} strongly structurally controllable pairs $(\mathbf{A},\mathbf{B})$ that satisfy Assumptions (A1)-(A3).

Proof: As a consequence of Theorem 3.1, the lower triangular part of the network adjacency matrix contains exactly $n-m$ fixed columns. These columns satisfy $\text{supp}(\mathbf{A}(i:\text{end}, j)) = \{1\}$, $i = m+1,\dots,n$, $j \in \{1,\dots,n-1\}$, and have lengths $\{n-m,\dots,1\}$. The remaining $m-1$ columns are not constrained. Let p_i be the position of the column with fixed structure and length i. Then, $p_{n-m} \in \{1,\dots,m\}$, $p_{n-m-1} \in \{1,\dots,m+1\} \setminus \{p_{n-m}\}$, and, recursively, $p_{n-m-j} \in \{1,\dots,m+j\} \setminus \{p_{n-m},\dots,p_{n-m-j}\}$. Thus, each column with fixed structure can be independently positioned in a set of m locations, and the statement follows. ■

Lemma 4.1 provides an intuitive, yet conservative, lower bound on the number of strongly structurally controllable networks. Indeed, the bound is obtained by counting the number of networks with exactly $n-m$ edges (excluding self-loops) that are strongly structurally controllable. An illustration of this procedure is in Fig. 2. Lemma 4.1 can also be used to characterize the ratio of strongly structurally controllable networks over the total number of networks. To this aim, let $\mathcal{N}$ be the set of all networks with n nodes, and let $\Omega_m \subseteq \mathcal{N}$ be the set of strongly structurally controllable networks with m control nodes.

Notice that $|\mathcal{N}| = 2^{n(n-1)}$ due to Assumption (A1). We now present the main result of this paper.

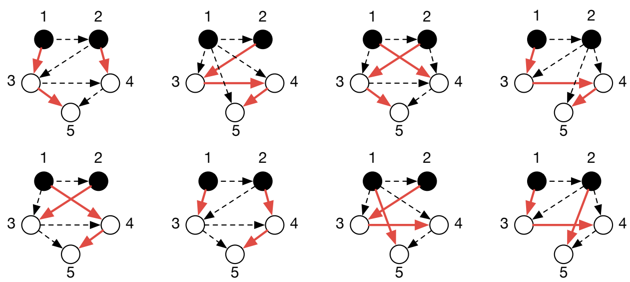


Fig. 2. The figure shows all possible SSC networks with 5 nodes and 2 control nodes. Red edges are used to depict networks with exactly $n-m$ edges, according to Lemma 4.1. Notice that there exist m^{n-m} different arrangements of these 3 edges. Dashed edges, as well as all edges represented by upper triangular entries of the adjacency matrix (omitted here for the sake of clarity), denote the existence of a set of SSC networks for any combination of the fixed structure columns.

Theorem 4.2: ***(Bounds on the number of SSC networks)*** For every $n, m \in \mathbb{N}$,

$$\underline{\alpha} \le \frac{|\Omega_m|}{|\mathscr{N}|} \le \overline{\alpha},$$

where

$$\underline{\alpha} = 2^{\frac{1}{2}\left(-n^2+n+m^2-m\right)} \sum_{j=1}^{m} 2^{(n-m)(m-j)}(2^{n-m}-1)^{j-1},$$
$$\overline{\alpha} = 2^{\frac{1}{2}\left(-n^2-n-m^2+m\right)+nm} m^{m-n}. \quad (3)$$

Proof: Let $\mathbf{A}$ be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad (4)$$

where $\mathbf{A}_{11} \in \mathbb{R}^{m\times m}$, $\mathbf{A}_{12} \in \mathbb{R}^{m\times n-m}$, $\mathbf{A}_{21} \in \mathbb{R}^{n-m\times m}$, and $\mathbf{A}_{22} \in \mathbb{R}^{n-m\times n-m}$.

Suppose that $\text{supp}(\mathbf{A}(i : \text{end}, i-1)) = \{1\}$, $i = m+2, \ldots, n$, while the tallest column with a fixed structure is on the left-most side of $\mathbf{A}_{21}$, namely $\mathbf{A}(m+1 : n,\ 1)$. We count the number of strongly structurally controllable networks by enumerating all admissible networks obtained by fixing the remaining $(n-m)(m-1)$ free entries left in $\mathbf{A}_{21}$. Suppose now that the tallest fixed column is $\mathbf{A}(m+1 : n,\ 2)$. Notice that the free entries on its right-hand side are $(n-m)(m-2)$, whereas the free entries on the left-hand side can be arranged in $(2^{n-m}-1)$ possible ways, i.e. we are removing the realizatio with $\text{supp}(\mathbf{A}(m+1 : n,\ 1)) = \{1\}$ that has been counted already. Suppose now that the tallest fixed column is $\mathbf{A}(m+1 : n,\ 3)$. The free entries to its right-hand side are exactly $(n-m)(m-3)$, to which we add the entries on the left omitting realizations with unitary support. This procedure is repeated until the tallest fixed column is $\mathbf{A}(m+1 : n,\ m)$. According to this procedure, $\underline{\alpha}$ is

$$\frac{2^{\frac{1}{2}(n(n-1)+m(m-1))} \sum_{j=1}^{m} 2^{(n-m)(m-j)}(2^{n-m}-1)^{j-1}}{|\mathscr{N}|}.$$

Yet, Ω_m clearly comprises different positions of *all* the fixed columns, hence the first inequality holds.

To prove the upper bound, we recall that, according to Lemma 4.1 , there are m^{n-m} possible arrangements for the fixed structure columns. The remaining arbitrary entries in the lower part of $\mathbf{A}_{22}$ and in $\mathbf{A}_{21}$ are $(n-m)(m-1)$, independently of the arrangement of the fixed structure columns. Therefore, $\overline{\alpha}$ is obtained by simplifying

$$\frac{m^{n-m} 2^{\frac{1}{2}(n(n-1)+m(m-1))+(n-m)(m-1)}}{|\mathscr{N}|}.$$

The second inequality holds because, by arranging all the possible combinations of free entries left for each one of the m^{n-m} placements of fixed structure columns, we count multiple times matrices (networks) that have already been considered with different placements. ■

Theorem 4.2 implies that (i) if the number of control nodes depends linearly (with rate smaller than 1) on the network cardinality, then the fraction of strongly structurally controllable networks decreases to zero as the network cardinality increases; also, (ii) if the number of control nodes grows with the same rate as the network cardinality, then the fraction of strongly structurally controllable networks remains bounded as the network cardinality increases. These results are formalized in the following corollary.

Corollary 4.3: ***(Limiting bounds with a fraction of control nodes)*** If $m \le \beta n$, with $0 < \beta < 1$, then,

$$\lim_{n\to\infty} \frac{|\Omega_m|}{|\mathscr{N}|} = 0.$$

Moreover, if $m = n-k$, for some constant $k \in \mathbb{N}$ (independent of n), then

$$\lim_{n\to\infty} \frac{|\Omega_m|}{|\mathscr{N}|} > 0.$$

Proof: Notice that, as $n \to \infty$ asymptotically we have $\frac{|\Omega_m|}{|\mathscr{N}|} \le \overline{\alpha} \approx \frac{2^{n^2} n^n}{2^{2n^2}} = \frac{n^n}{2^{n^2}}$. To prove that this ratio converges to 0 as the cardinality increases, we show that $\log_2(\frac{2^{n^2}}{n^n}) \to \infty$. Indeed, $\log_2(\frac{2^{n^2}}{n^n}) = n^2 - n\log_2(n) \to \infty$.

To prove the second statement and the convergence of the limit when $m = n-k$, it follows from Theorem 4.2 that

$$\frac{|\Omega_m|}{|\mathscr{N}|} \ge \underline{\alpha} = \frac{\sum_{j=1}^{n-k} 2^{k(n-k-j)}(2^k-1)^{j-1}}{2^{(n-k)k} 2^{k(k-1)/2}} = \frac{\frac{1}{2^k}\sum_{j=1}^{n-k}\left(\frac{2^k-1}{2^k}\right)^{j-1}}{2^{k(k-1)/2}}.$$

Notice that $\frac{2^k-1}{2^k} < 1$. Thus, for $n \to \infty$ we have

$$\sum_{j=1}^{\infty}\left(\frac{2^k-1}{2^k}\right)^{j-1} = \sum_{j=0}^{\infty}\left(\frac{2^k-1}{2^k}\right)^{j} = \frac{1}{1-\frac{2^k-1}{2^k}},$$

and it follows

$$\frac{|\Omega_m|}{|\mathscr{N}|} \ge \underline{\alpha} = \frac{\frac{1}{2^k}\frac{1}{1-\frac{2^k-1}{2^k}}}{2^{k(k-1)/2}} = \frac{1}{2^{k(k-1)/2}}.$$

■

Fig. 3 and 4(a) show the behavior of $\underline{\alpha}$ and $\overline{\alpha}$ and the results in Corollary 4.3. The following Lemma improves the bound in Theorem 4.2, and shows that the lower bound $\underline{\alpha}$ is guaranteed to converge to 1 for $k = 2$.

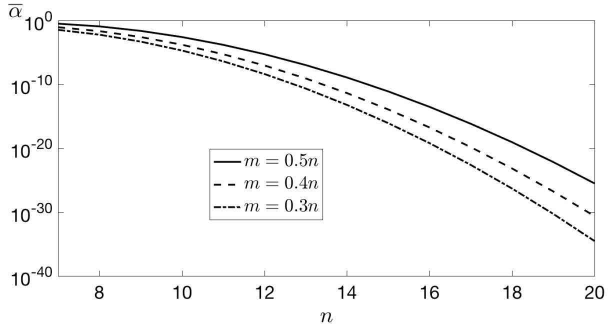
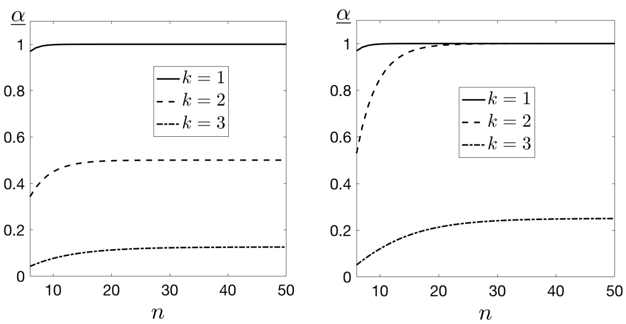


Fig. 3. This figure shows in a semilogarithmic plot that, when $m \leq \beta n$, $\lim_{n\to\infty} \overline{\alpha} = 0$. Thus, because of Theorem 4.2, the ratio between the number of SSC networks and the number of possible networks with the same cardinality and number of control nodes tends to zero.



(a) (b)

Fig. 4. (a) The figure shows that if $m = n - k$, for some constant $k \in \mathbb{N}$, then $\lim_{n\to\infty} \underline{\alpha} > 0$. (b) The figure depicts the improvements in the convergence value of $\underline{\alpha}$ when γ is added, according to Corollary 4.4.

Lemma 4.4: ***(Improved lower bound)*** Let $\underline{\alpha}$ be as in (3), then

$$\underline{\alpha} + \gamma \leq \frac{|\Omega_m|}{|\mathcal{N}|},$$

where

$$\gamma = 2^{\frac{1}{2}(n(n-1)+m(m-1))} \sum_{j=2}^{m-1} \left[\sum_{i=1}^{j-1} \binom{j}{i}\right] \binom{m}{j} (2^k - 2)^{m-j}.$$

Proof: Let $\mathbf{A}$ be partitioned as in (4). Assume that all the fixed structure columns are on the right-most side of the lower triangular part of $\mathbf{A}$. Now, suppose that the shortest one (one entry) is moved from its position and the entry $\mathbf{A}(n, n-1)$ is fixed to zero. The term $\sum_{j=2}^{m-1} \binom{n-2}{j}(2^k - 2)^{m-j}$ represents all the possible arrangements of columns $[\times \ \dots \ \times]^\mathsf{T}$ and $[0 \ \dots \ 0]^\mathsf{T}$ among the arbitrary columns of $\mathbf{A}_{21}$. This term must be multiplied by $\sum_{i=1}^{j-1} \binom{j}{i}$, which counts the positionings of columns $[\times \ 0 \ \dots \ 0]^\mathsf{T}$ and $[0 \ \dots \ 0 \ \times]^\mathsf{T}$. Recall that they must appear at least once for Theorem 3.1 to be respected. ■

Fig. 4(b) depicts the improved lower bound, showing better convergence values with respect to the lower bound $\underline{\alpha}$. Different column structures can be considered to improve this bound further, but the downside is that complex combinatorial terms must be taken into account.

V. Conclusion

This paper presents an algebraic characterization of strongly structurally controllable networks with self loops. The presented condition relies on control and graph theoretic notions, and leads to the design of an efficient algorithm to test for strong structural controllability of a network. This result is further exploited to enumerate the number of strongly structurally controllably networks, and to show that (i) if the number of control nodes depends linearly (with rate smaller than 1) on the network cardinality, then the fraction of strongly structurally controllable networks decreases to zero as the network cardinality increases, and (ii) if the number of control nodes grows with the same rate as the network cardinality, then the aforementioned fraction remains bounded as the network cardinality increases.

VI. Appendix

In order to prove Theorem 3.1, we will first present the following result on partitioned systems controllability.

Lemma 6.1: ***(Controllability of partitioned systems)*** Let the pair (A, B) be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times (n-n_1)}$, $A_{21} \in \mathbb{R}^{(n-n_1)\times n_1}$, $A_{22} \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$, and $B_1 \in \mathbb{R}^{n_1 \times m}$. The pair (A, B) is controllable only if (A_{22}, A_{21}) is controllable.

Proof: Notice that the partitioned dynamics read as

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t). \quad (5)$$

System (5) is controllable only if the following auxiliary system is controllable:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{u}(t). \quad (6)$$

Let $\bar{u}(t) = v(t) - A_{11}x_1(t) - A_{12}x_2(t)$, and notice that (6) is controllable if and only if the following system is controllable:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{\bar{B}} v(t).$$

The controllability matrix of the pair $(\bar{A}, \bar{B})$ is

$$\begin{aligned} \mathcal{C}(\bar{A}, \bar{B}) &= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & A_{21} & A_{22}A_{21} & \cdots & A_{22}^{n-1}A_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathcal{C}(A_{22}, A_{21}) \end{bmatrix}, \end{aligned}$$

from which we conclude that (6) is controllable only if the controllability matrix $\mathcal{C}(A_{22}, A_{21})$ of the pair (A_{22}, A_{21}) is full rank, which concludes the proof. ■

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: *(i) ⇒ (ii)*. Given $i \in \{m+1, \dots, n\}$, let $\mathbf{A}$ and $\mathbf{B}$ be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix},$$

where $\mathbf{A}_{11} \in \mathbb{R}^{i-1 \times i-1}$, $\mathbf{A}_{12} \in \mathbb{R}^{i-1 \times n-i+1}$, $\mathbf{A}_{21} \in \mathbb{R}^{n-i+1 \times i-1}$, and $\mathbf{A}_{22} \in \mathbb{R}^{n-i+1 \times n-i+1}$. We proceed by contradiction.

Assume that, for some $i \in \{m+1, \ldots, n\}$, $|\text{supp}\{\mathbf{A}(i : \text{end}, j)\}| \neq 1$ for all $j \in \{1, \ldots, i-1\}$. Let $\mathbb{1}$ be the vector of all ones, and consider a realization (A, B) where $\mathbb{1}$ is an eigenvector of A_{22}, and $A_{21}^{\mathsf{T}} \mathbb{1} = 0$. Notice that such a realization exists because of Assumption (A1), that is, each row of $\mathbf{A}_{22}$ contains at least one nonzero entry, and because $|\text{supp}\{\mathbf{A}(i : \text{end}, j)\}| > 1$. For instance, select A_{22} so that all its rows sum up to 1, and A_{21} so that all its columns sum up to zero. From the eigenvector test [28], the pair (A_{22}, A_{21}) is not controllable. Finally, from Lemma 6.1 (A, B) is also not controllable, which contradicts our initial assumption.

From the above reasoning we conclude that, if $(\mathbf{A}, \mathbf{B})$ is strongly structurally controllable, then for every $i \in \{m+1, \ldots, n\}$ there exists $j \in \{1, \ldots, i-1\}$ such that $|\text{supp}\{\mathbf{A}(i : \text{end}, j)\}| = 1$. Assume now that $\text{supp}\{\mathbf{A}(i : \text{end}, j)\} \neq \{1\}$ for some $i \in \{m+1, \ldots, n\}$ and for all $j \in \{1, \ldots, i-1\}$. Let h be the first index such that $\text{supp}\{\mathbf{A}(h : \text{end}, j)\} = \{k\}$, with $k > 1$, and let $\tilde{\mathbf{A}}$ be the matrix obtained from $\mathbf{A}$ by switching the h-th and k-th rows and columns. Notice that $\text{supp}\{\tilde{\mathbf{A}}(i : \text{end}, j)\} = \{1\}$ for all $i \in \{m+1, \ldots, h\}$ and for some $j \in \{1, \ldots, i-1\}$. Further, $(\mathbf{A}, \mathbf{B})$ is permutation-similar to $(\tilde{\mathbf{A}}, \mathbf{B})$. By iterating this procedure two cases are possible: (1) condition (ii) in Theorem 3.1 is satisfied, that is, $\text{supp}\{\mathbf{A}(i : \text{end}, j)\} = \{1\}$ for all $i \in \{m+1, \ldots, n\}$ and for some $j \in \{1, \ldots, i-1\}$, or (2) $|\text{supp}\{\tilde{\mathbf{A}}(i : \text{end}, j)\}| \neq 1$ for some $i \in \{h+1, \ldots, n\}$ and for all $j \in \{1, \ldots, i-1\}$, which violates our initial assumption. We conclude that statement (i) implies (ii).

(ii) ⇒ (i). To prove the necessity of condition (ii) we employ the notions of *zero forcing set* and coloring rule. Namely, we show that we can generate a *chronological list of forces* that satisfies [23, Theorem 5.5]. According to statement (ii), for any $i \in \{m+1, \ldots, n\}$ there exists $j \in \{1, \ldots, i-1\}$ such that i is the only *white* out-neighbor of j; thus j *forces* i. Also, since $i > j$, the list does not contain any force of the form $i \to i$. Thus, $\{1, \ldots, m\}$ is a zero forcing set of $\mathcal{G}_{\tilde{\mathbf{A}}}$. We conclude that $(\tilde{\mathbf{A}}, \mathbf{B})$ is strongly structurally controllable, and so is $(\mathbf{A}, \mathbf{B})$ because permutation-similar to $(\tilde{\mathbf{A}}, \mathbf{B})$. ■

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