2017 American Control Conference
Sheraton Seattle Hotel
May 24–26, 2017, Seattle, USA

Time-Invariant versus Time-Varying Actuator Scheduling in Complex Networks

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Abstract— **This paper studies the benefits of time-varying actuator scheduling to the controllability of complex networks. The network dynamics are described by a single-input discrete-time linear system over an undirected graph. Taking the trace of the controllability Gramian as the measure of network controllability, we identify a new notion of nodal communicability and unveil its role in the time-varying actuator scheduling problem. We then proceed to identify conditions on the network structure that determine whether time-varying actuator scheduling is better than time-invariant actuator selection. The main conclusion of our results is that having several and heterogeneous central nodes (versus having a single highly central node) is the common factor in networks where time-varying actuator scheduling is advantageous.**

I. Introduction

After its emergence in social sciences in the mid-twentieth century, the field of complex networks has attracted extensive research in multiple disciplines, including physics, computer science, neuroscience, and systems theory. Given the universal ubiquity of complex networks, there is a need to develop systematic ways to analyze their behavior, understand how individual components affect their overall response, and characterize their robustness properties. This work contributes to this body of work by exploring the advantages of time-varying actuator scheduling in complex networks.

Literature review: The literature on complex networks is vast and diverse, cf. [1]–[3] and references therein. Our work builds on the growing literature on controllability of complex linear networks, which seeks to address two fundamental questions: how network controllability relates to "macroscopic" network properties such as size, degree distribution, etc., [4]–[7], and how to choose the "best" set of control nodes [8]–[10] to minimize actuation energy. In addressing these questions, the use of binary controllability measures [4], see also [5], [8], is oblivious to the "difficulty" (in terms of energy cost) of steering the network in different directions in the state space. This has motivated the introduction of a number of controllability metrics to quantify the control effort, including the smallest eigenvalue, determinant, and trace of the Gramian [6], [7], [9], [10]. The recent work [11] characterizes the (sub-)modularity property of many Gramian-based measures. With the exception of [10], these works build on the assumption that the set of control nodes is fixed over time. Depending on the specific network structure, this might impose a significant limitation on its controllability, especially for large-scale systems. Instead, [10] designs various algorithms for time-varying actuator scheduling to optimize the smallest eigenvalue of the Gramian as a metric of the worst-case energy required to steer the network to a desired state. However, this work does not study the fundamental network properties that make time-varying actuation selection beneficial. Here we seek to address this question using the trace of the Gramian as a metric for the average energy required to steer the network in all directions in the state space.

Statement of contributions: We consider complex networks whose dynamics are described by a single-input discrete-time linear system over an undirected graph. As measure of network controllability, we consider the trace of the Gramian and examine the extent to which time-varying actuation selection is beneficial. Our first contribution is the introduction of a new notion of nodal communicability, termed $2k$-communicability. We show how the optimal time-varying actuator scheduling problem can be fully encoded in terms of this notion: at each time index, the input should be actuated at the node with largest communicability. We also provide an upper bound on the number of control node switches as the time index grows. Our second contribution identifies three types of conditions on the network structure guaranteeing that time-invariant actuator selection is optimal. The common factor in these conditions is the existence of a highly distinct authority in the network, i.e., a single node with distinctly higher influence on the dynamics. We show that uniform line, ring, and star networks without self-loops as well as Barabási-Albert scale-free random networks (with high probability) belong to this group of networks. In contrast, our third contribution establishes conditions on the network structure under which time-varying actuator scheduling is indeed optimal. Our results show that the determining factor in making time-varying actuation beneficial is the existence of *multiple heterogeneous* central nodes in the network. We show that networks with small but "powerful" (in terms of link weights) subnetworks as well as Watts-Strogatz small-world networks (with high probability) belong to this group of networks, whereas Erdös-Rényi random networks do not (with high probability). Due to space constraints, proofs are omitted and available at [12].

II. Preliminaries

This section introduces the notation and reviews basic concepts on graph theory and network centrality.

1) Notation: We use $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{N}$, and $\mathbb{E}$ to denote the set of reals, integers, non-negative integers, positive integers, and positive even integers, respectively. For $a, b \in \mathbb{Z}$, $a|b$ denotes that a divides b. The n-vector of all ones is denoted by $\mathbf{1}_n$ and $\{e_i\}_{i=1}^n$ stands for the standard basis of $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, x_i and $(x)_i$ refer to its ith component. Similarly, a_{ij} and $(A)_{ij}$ refer to the (i, j)'th entry of a matrix A. For

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$\lambda \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}_{\geq 0}$, $\lambda^\ell \triangleq [\lambda_1^\ell \ \cdots \ \lambda_n^\ell]^T$. We use bold-face letters for finite sequences of the form $\boldsymbol{u}_K \triangleq (u(k))_{k=0}^{K-1}$ and use the notation $\|\boldsymbol{u}_K\|_F$ for $\big(\sum_{j=0}^{K-1} u(k)^2\big)^{1/2}$. Given a matrix $M \in \mathbb{R}^{n \times n}$, its trace, determinant, rank, and its eigenvalue with smallest magnitude are denoted by $\mathrm{tr}(M)$, $|M|$, $\mathrm{rank}(M)$, and $\lambda_{\min}(M)$, respectively. For two functions $f, g : \mathbb{N} \to \mathbb{R}$, $f(n)$ is $\mathcal{O}(g(n))$ if there exist $C \geq 0$ and $N \in \mathbb{N}$ such that $f(n) \leq Cg(n)$ for $n \geq N$. A matrix V is orthogonal if $V^{-1} = V^T$. A nonnegative matrix is doubly-stochastic if all of its rows and columns sum up to one.

2) Graph Theory: An undirected graph $\mathcal{G} = (V, E)$ consists of a set $V = \{1, \ldots, n\}$ of nodes and a set $E = \{\{i, j\} \mid i \text{ is connected to } j\}$ of edges. A weighted undirected graph $\mathcal{G} = (V, E, A)$ also includes an adjacency matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ where for any $i, j \in V$, $a_{ij} \geq 0$ is the weight of the edge between nodes i and j (including self-loops). A path in $\mathcal{G}$ from node i to j is a finite sequence $\ell_0, \ell_1, \ldots, \ell_p$ of nodes where $\ell_0 = i$, $\ell_p = j$, and $\{\ell_{m-1}, \ell_m\} \in E$ for $\ell \in \{1, \ldots, p\}$. A cycle is a path with $\ell_0 = \ell_p$. For $k \geq 1$, $(A^k)_{ij}$ gives the (weighted) number of paths of length k between nodes i and j.

3) Network Centrality: Consider a network of size n represented by a graph $\mathcal{G} = (V, E, A)$. In this network, some nodes may have a large influence on the rest of the network, while others are mostly "followers". Centrality notions [13]–[16] are ways of quantifying the different roles, in terms of influence, among the nodes. Here, we briefly review three centrality measures with spectral characterizations.

a) Eigenvector Centrality: Let $v \in \mathbb{R}_{\geq 0}^n$ be the vector of centrality values of all nodes. Eigenvector centrality [15], [17] is based on the idea that the influential nodes are the ones that are connected to other influential nodes. A vector (and usually the only vector) v that satisfies this condition is the eigenvector corresponding to the largest eigenvalue of A, which is thus *defined* as the vector of eigenvector centralities. Throughout the paper, unless otherwise noted, "centrality" refers to eigenvector centrality.

b) Exponential and Resolvent Communicability: Different notions of communicability have been proposed for complex networks. For a given node i, these include the exponential communicability $(e^{\beta A})_{ii}$ and the resolvent communicability $((I - \beta A)^{-1})_{ii}$, respectively, where $\beta > 0$ is a parameter. Having the power series of $e^{\beta A}$ and $(I - \beta A)^{-1}$ in mind, it follows that the exponential and resolvent communicabilities count the total number of cycles that path through node i, weighting the "importance" of cycles of length k by $\beta^k/k!$ and β^k, respectively. Therefore, the role of β is to determine how local or global these measures are: increasing β increases the weights of longer cycles. One can show [14] that in the extreme cases of $\beta \to \infty$ for the exponential and $\beta \to \frac{1}{\lambda_{\max}(A)}$ in the resolvent case, both notions result in the same rankings of the nodes as eigenvector centrality.

c) Degree Centrality: The degree centrality of a node i is the sum of the i'th row (or column) of A.

III. Problem Statement

We consider a network of n nodes that communicate, in discrete time, over an undirected graph $\mathcal{G}$ with adjacency matrix A. We assume each node has linear and time-invariant dynamics and that at each time, one node can be controlled exogenously, resulting in the overall network dynamics

$$x(k+1) = Ax(k) + b(k)u(k), \qquad k \in \mathbb{Z}_{\geq 0}. \tag{1}$$

Here, $x_i(k) \in \mathbb{R}$ is the state of node i for $i \in \{1, \ldots, n\}$, $u(k) \in \mathbb{R}$ is the control input, and $b(k)$ is the time-varying input matrix, all at time k. Since we can always normalize $b(k)$ and include its magnitude in $u(k)$, we assume $\|b(k)\| = 1$ so $b(k) \in \{e_i\}_{i=1}^n$.

The controllability of the system (1) at time $K \in \mathbb{N}$ is defined as the ability to steer the state of the network from any initial condition $x(0)$ to any desired state $x(K)$ at time K. It is well-known [18] that (1) is controllable at time K if and only if the controllability Gramian, namely,

$$\mathcal{W}_K \triangleq \sum_{k=0}^{K-1} A^k b(K{-}1{-}k) b(K{-}1{-}k)^T (A^T)^k, \tag{2}$$

is nonsingular. Since the network is undirected, $A^T = A$. It is also well-known [18] that if (1) is controllable at time K, among all the controls $\boldsymbol{u}_K \triangleq (u(k))_{k=0}^{K-1}$ that can steer the network from the origin to an arbitrary $x_f \in \mathbb{R}^n$ at time K, the one with minimum energy $\|\boldsymbol{u}_K\|_F$ is given by

$$u^*(k) = b(k)^T (A^T)^{K-1-k} \mathcal{W}_K^{-1} x_f, \quad k \in \{0, \ldots, K-1\}.$$

It is immediate to verify that $\|\boldsymbol{u}_K^*\|_F^2 = x_f^T \mathcal{W}_K^{-1} x_f$. It is thus desirable to have $\mathcal{W}_K^{-1}$ as "small" as possible, or equivalently, $\mathcal{W}_K$ as "large" as possible. To quantify how large the Gramian is, several spectral measures have been proposed in the literature [7], [11], [19], including $\lambda_{\min}(\mathcal{W}_K)$, $|\mathcal{W}_K|$, $\mathrm{tr}(\mathcal{W}_K^{-1})$, $\mathrm{tr}(\mathcal{W}_K)$, and $\mathrm{rank}(\mathcal{W}_K)$. While each metric has advantages and disadvantages, we focus here on $\mathrm{tr}(\mathcal{W}_K)$ due to its linearity and tractability. This metric is inversely related to the average energy required to steer the network in all directions in the state space, thus characterizing the average controllability of the network.

Accordingly, we are interested in the solution of the following optimization problem:

$$\boldsymbol{b}_K^* = \underset{\boldsymbol{b}_K \in \mathcal{F}_{tv}}{\arg\max} \ \mathrm{tr}(\mathcal{W}_K), \tag{3}$$

where $\mathcal{F}_{tv} = \{e_1, \ldots, e_n\}^K$ is the feasible set of *time-varying* input matrices. Using the definition (2) and the invariance of trace under cyclic permutations, we can write

$$\mathrm{tr}(\mathcal{W}_K) = \sum_{k=0}^{K-1} b(K{-}1{-}k)^T A^{2k} b(K{-}1{-}k).$$

Therefore, since $b(K{-}1{-}k)^T A^{2k} b(K{-}1{-}k) = (A^{2k})_{ii}$ where i is the index of the node to which $u(K-1-k)$ is applied, the optimization at each time $K{-}1{-}k$ boils down to finding the largest diagonal element of A^{2k}.

Note that the computation of the exact solution to (3) is feasible (with polynomial time complexity) for large networks since the optimization in (3) is completely decoupled over time. This is because the $\mathrm{tr}(\mathcal{W}_K)$ is a modular set function, while the other measures mentioned above are submodular, for which greedy algorithms are usually employed.

If we constrain the actuator scheduling sequence $\boldsymbol{b}_K$ to be time invariant, then instead of (3) we have

$$\boldsymbol{b}_K^* = \arg\max_{\boldsymbol{b}_K \in \mathcal{F}_{ti}} \operatorname{tr}(\mathcal{W}_K),$$
$$\mathcal{F}_{ti} = \left\{\boldsymbol{b}_K \in \mathcal{F}_{tv} \mid b(1) = b(2) = \cdots = b(K-1)\right\}.$$

Since $\mathcal{F}_{ti} \subset \mathcal{F}_{tv}$, it is clear that in general the solution of this problem will be sub-optimal with respect to (3) and average controllability will be worse. However, a time-varying actuator scheduling is more difficult and/or expensive to implement in practice as it requires an actuator to be connected to more than one (ideally every) node in the network and a new optimization problem to be solved at each time instance. Therefore, it is important to determine under what conditions and for which networks, the optimal time-varying actuator scheduling will outperform the optimal time-invariant one. This problem is the focus of our work. The main message is that, in general, networks with a single distinct authority (central node) will not benefit from time-varying actuator schedules while networks with many comparable, yet heterogeneous authorities will.

IV. $2k$-Communicability

In this section, we introduce the notion of $2k$-communicability and explain its connection with the optimal actuator scheduling problem. We also discuss its similarities and differences with the existing notions of communicability as well as its limiting scenarios where $k = 1$ and $k \to \infty$.

Let $A = V\Lambda V^T$ be the eigen-decomposition of A where $V = [v_{ij}]_{n\times n}$ is orthogonal and Λ is diagonal. Let λ_i be the i'th diagonal entry of Λ where $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and define $\lambda = [\lambda_1 \ \cdots \ \lambda_n]^T$. Let $W = [w_{ij}]_{n\times n}$ be the doubly stochastic matrix given by $w_{ij} = v_{ij}^2$ for all $i, j \in \{1, \ldots, n\}$. After some algebraic manipulations, it follows that $(A^{2k})_{ii} = (W\lambda^{2k})_i = \sum_{j=1}^n v_{ij}^2 \lambda_j^{2k}$, and the optimal input matrix can be written as

$$b^*(K-1-k) = e_{\arg\max_{1\leq i\leq n}(A^{2k})_{ii}}, \tag{4}$$

for all $k \in \{0, \ldots, K-1\}$. For each $i \in \{1, \ldots, n\}$, $(A^{2k})_{ii}$ is a convex sum of n exponential functions. The one function among $(A^{2k})_{11}, \ldots, (A^{2k})_{nn}$ which is on top at time k determines $b^*(K-1-k)$. Therefore, all we need to know is the number of times that the maximum of these n sums of n exponential functions change over $\{0, \ldots, K-1\}$. If the maximum does not change, a time-varying input allocation is not beneficial and vice versa.

As described in Section II, exponential and resolvent communicabilities count a weighted number of the total number of cycles *of all lengths* that pass through each node in the network. In the following, we define a similar notion that accounts for the paths of finite length.

Definition IV.1. *($2k$-communicability).* Consider a dynamic network of n nodes defined by the adjacency matrix A. For any $k \in \mathbb{N}$, the *$2k$-communicability* of each node $i \in \{1, \ldots, n\}$ is $R_i(k) = (A^{2k})_{ii}$. □

The $2k$-communicability of any node i counts the (weighted) number of cycles of exact length $2k$ that pass through node i. The advantage of this notion is its direct connection with optimal actuator placement in discrete-time networks, as (4) shows. Interestingly, the same role is played by the notion of exponential communicability in continuous-time networks (where β plays the "time" role of k). Also, a different but related notion of centrality is proposed in [9].

To study the number of changes in $\max_i R_i(k)$, it is sometimes convenient to extend the domain of $\{R_i\}_{i=1}^n$ to $\mathbb{R}_{\geq 0}$. For consistency, define $R_i(t) = (W\lambda^{2t})_i$ for $t \in \mathbb{R}_{\geq 0}$ and $i \in \{1, \ldots, n\}$. The following result, whose proof is straightforward, summarizes a few properties of this function.

Lemma IV.2. ***(Basic properties of R_i).*** *For $i \in \{1, \ldots, n\}$, $R_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is smooth and strictly convex, satisfies $R_i(0) = 1$, and is monotonically decreasing if $\lambda_1 \leq 1$.*

Figure 1 shows a small network of $n = 6$ nodes (without self-loops) as well as the evolutions of $\{R_i(t)\}_{i=1}^n$, where the optimal control node switches $n - 2 = 4$ times.

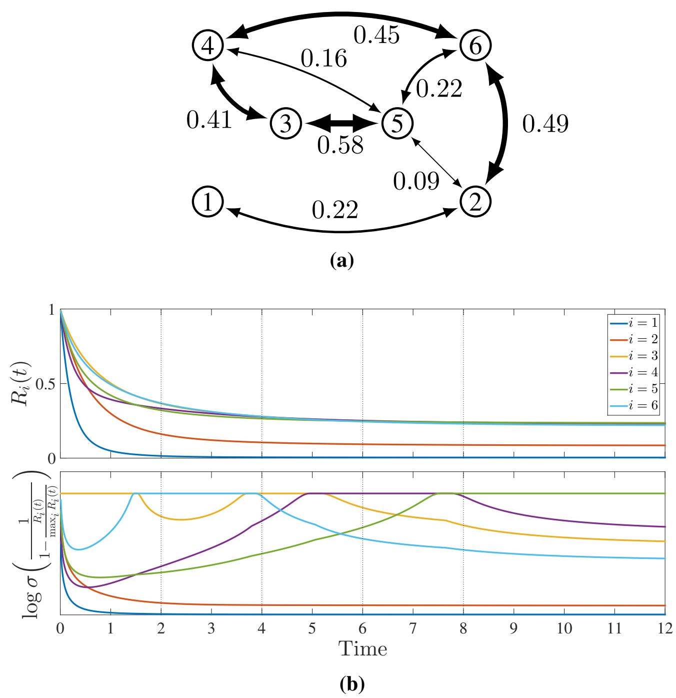

Fig. 1: Network of $n = 6$ nodes with $n-2$ changes in the optimal control node over time. (a) Network topology. The thickness of the edges is proportional to their weights. (b) The evolution of the functions $\{R_i\}_{i=1}^n$ (top) and a logarithmic function of them that preserves the order (bottom) – to better illustrate the switches between the curves ($\sigma : \mathbb{R} \to \mathbb{R}$ is a sigmoidal function).

For general networks, the following result provides an upper bound on the possible number of switches of $\arg\max_i R_i(t)$.

Lemma IV.3. ***(Bound on the number of control node switches).*** *The maximum possible number of switches in $\arg\max_{1\leq i\leq n} R_i(t)$ over $t \in \mathbb{R}_{\geq 0}$ is $\mathcal{O}(n^3)$, where n is the network dimension.*

Lemma IV.3 rules out the possibility of an arbitrarily large number of control node switches (however, as Examples V.3 and VI.2 show later, the bound $\mathcal{O}(n^3)$ is conservative). This result highlights how the optimal actuator scheduling is determined by the dependence of the nodes' $2k$-communicability on the time index k. For small k, this notion depends on the local network structure, and incorporates more global information as k grows. In particular,

(i) The 2-communicability of a node is related to its (weighted) degree centrality. To see this, note that $R_i(1) = (A^2)_{ii} = \sum_{j=1}^{n} a_{ij}^2$, so $R_i(1)$ is equal to the degree in unweighted networks. In the case of weighted networks, the 2-communicability and degree centrality become more different as the network weights become more heterogeneous.

(ii) The ∞-communicability of a node, i.e., $(A^{2k})_{ii}$ as $k \to \infty$, results in the same ordering of the nodes as the square of the eigenvector centrality (assuming that $|\lambda_1| > |\lambda_2|$). This follows from $\lim_{k\to\infty} A^{2k} = v_1 v_1^T$, where v_1 is the vector of centralities if $\lambda_1 = 1$. If $\lambda_1 \neq 1$, we can either take k large enough or normalize A by λ_1 and then take the limit $\lim_{k\to\infty} A^{2k}$ (which does not affect the node order).

In the remainder of the paper, we assume for simplicity that the largest element of the first column of W is w_{11}, i.e.,

$$w_{11} = \max_{1\leq i\leq n} w_{i1}. \tag{5}$$

This assumption can always be realized by a permutation of the rows of W, which corresponds to (re-)labeling the node with the highest eigenvector centrality as node 1.

V. Networks Where the Optimal Actuator Schedule is Time-Invariant

Here, we give conditions and examples of networks that do *not* benefit from time-varying actuator scheduling. The following result characterizes three such cases.

Theorem V.1. ***(Networks with a single extreme authority).*** *If any of the following conditions holds:*
(i) $\frac{1-w_{11}}{w_{11}} \leq \frac{|\lambda_1|-|\lambda_2|}{|\lambda_1|-|\lambda_n|}$,
(ii) $w_{11} + w_{12} = 1$,
(iii) the network has three or fewer nonzero eigenvalues with different absolute values and $1 \in \arg\max_i R_i(1)$,
then, for all $k \in \{0, \ldots, K-1\}$, $1 \in \arg\max_{1\leq i\leq n} R_i(k)$.

We next interpret the conditions in Theorem V.1:

• *(i)* holds for networks where there is a sufficiently distinct authority, in the sense of eigenvector centrality, and the network dynamics is dominated by the largest eigenvalue. Note that an extreme case of such networks is a totally disconnected network where $W = I$ and the highest authority is the node with the largest self-loop.

• *(ii)* holds for networks where the centrality of all nodes is determined by the weight of the link to the most central node. To see this, note that we have $w_{1j} = 0$ for $j \geq 3$, implying $v_{1j} = 0$ for $j \geq 3$. Since the rows of V are orthogonal, we deduce $v_{i2} = \alpha v_{i1}$ for all $i \geq 2$, where $\alpha = -v_{11}/v_{12}$ is constant. Using $A = V\Lambda V^T$, we have

$$a_{1i} = \lambda_1 v_{11} v_{i1} + \lambda_2 v_{12} v_{i2} = (v_{11}\lambda_1 + \alpha v_{12}\lambda_2) v_{i1},$$

so $v_{i1} \propto a_{1i}$ for all $i \geq 2$. The extreme case of such networks (as Example V.2 below shows) is the star network with no (or small-weight) self-loops.

• Regarding *(iii)*, the most well-known families of networks with three distinct eigenvalues are the complete bipartite networks and connected strongly regular networks. Moreover, cones on (n, k, λ, μ)-strongly regular graphs satisfying $\lambda_{\min}(A)(\lambda_{\min}(A) - k) = n$ are also known to have three distinct eigenvalues [20]. The other condition $1 \in \arg\max_i R_i(1)$ holds when the most (EV) central node has the largest 2-communicability (cf. the correlation between 2-communicability and degree in Section IV). The simplest example of a network with these properties is the star network (with no or equal self-loops for every node).

Example V.2. ***(Uniform line, ring, and star, networks).*** In the case of uniform line, ring, and star networks, cf. Figure 2, the values of $R_i(k)$ can be computed analytically, as given in Propositions I.1-I.3 in Appendix I. In all cases, we assume uniform edge and self-loop weights across the network (but the edge and self-loop weights need not be equal).

Line networks: the value of $R_i(k)$ increases with i until $i = \lceil \frac{n}{2} \rceil$ (i.e., the middle node) for $k \leq \lceil \frac{n}{2} \rceil - 1$ (this can be observed from the expression (7)). For general k, it can be shown that the value of the sum in (6) for $R_i(k)$ is strongly dominated by the summand corresponding to the index $p = 0$, which increases with i until $i = \lceil \frac{n}{2} \rceil$ and decreases afterwards. Thus, the optimal actuator scheduling is always to (one of) the center node(s), i.e., $b^*(k) = e_{\lceil \frac{n}{2} \rceil}$ for all k. If nodes have uniform self-loops, $R_i(k)$ can no longer be computed analytically but simulations show the exact same behavior as above.

Ring networks: the value of $R_i(k)$ does not depend on i (as shown by (8)), so the optimal actuator scheduling is arbitrary for all k. Similar result can be proved analytically if the nodes have uniform self-loops.

Star networks: if all self-loop weights are the same ($l_c = l_p$ in (9)), then $R_1(1) > R_i(1)$ for all $i \geq 2$ from (10). Therefore Theorem V.1(iii) implies that the center node is the optimal control node at all times. □

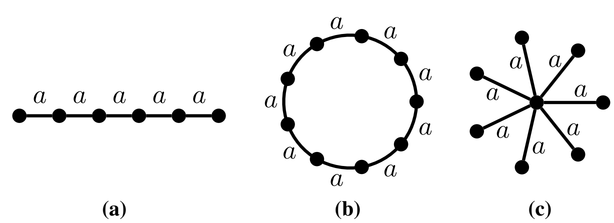

Fig. 2: Networks in Example V.2: (a) line network, (b) ring network, and (c) star network.

Example V.3. ***(Role of leader multiplicity and heterogeneity: Barabási-Albert scale-free networks).*** The takeaway message of Theorem V.1 is that time-invariant actuator scheduling is optimal for networks with clear distinct authorities. By construction, scale-free networks generated by the preferential attachment in [21] have this property with high probability, so we expect that they also have a time-invariant optimal actuator scheduling with high probability. Let p_{TV} be the probability of having at least one change in $\arg\max_i R_i(k)$. Figure 3 shows a heat map of p_{TV} as a function of the network size n and the number of links m added at each stage of the algorithm. For linear preferential attachment ($m = 1$), p_{TV} is almost zero irrespective of n, confirming the intuition above. For larger m, we observe a slow increase of p_{TV} as the network size grows. This is because for $m \geq 2$, more than one node receive a new link at each stage of the network growth, helping with the formation of multiple central nodes. This, however, does not automatically imply that these nodes will also be heterogeneous, which is why p_{TV} does not grow significantly. □

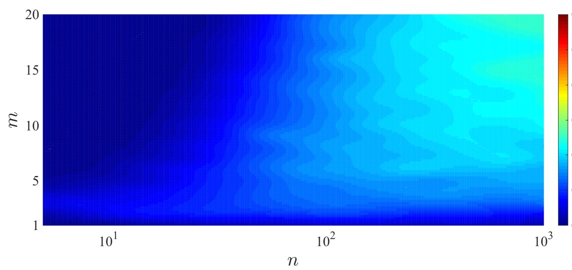

Fig. 3: The heat map of the probability of having at least one switch in $\arg\max_i R_i(k)$ as a function of network size n and link attachment rate m for Barabási-Albert scale-free networks.

VI. Networks Where the Optimal Actuator Schedule is Time-Varying

In this section, we give results and examples of the networks where the optimal actuator scheduling involves at least one change in the control node. The following result parallels Theorem V.1 and characterizes a class of networks where time-invariant actuator scheduling is *not* optimal.

Theorem VI.1. ***(Networks with heterogeneous authorities).*** *If $|\lambda_1| > |\lambda_2|$ and $R_i(1) > R_1(1)$ for some $i \in \{2, \ldots, n\}$ that has $v_{i1} < v_{11}$, then the optimal actuator scheduling involves more than one node when K is sufficiently large.*

The condition $|\lambda_1| > |\lambda_2|$ is not restrictive as it holds, by the Perron-Frobenius theorem, for all connected and aperiodic networks (recall that aperiodicity is in particular satisfied by the existence of any self-loops), cf. [22]. As we mentioned in Section V, the 2-communicability of a node is closely related to the degree centrality if the weights are all in the same range. For such networks, the condition $R_i(1) > R_1(1)$ requires that the nodes with highest eigenvector and degree centralities do not coincide, preventing the existence of extreme network authorities. This is, for instance, the case in the network of Figure 1(a).

An important take-away message from Theorem VI.1 is that for a change to occur in $\arg\max_i R_i(k)$, besides the existence of multiple leaders, heterogeneity of leaders is also necessary (a property that, e.g., a uniform ring network lack). In other words, for time-varying actuator scheduling to be beneficial, some node(s) should have the most local significance (to maximize $R_i(k)$ for small k) while *different* node(s) have global centrality (to maximize $R_i(k)$ for large k). The following example illustrates this point.

Example VI.2. ***(Role of leader multiplicity and heterogeneity, cont'd: Erdös-Rényi and Watts-Strogatz networks).*** Figure 4 shows a heat map of the probability p_{TV} of having at least one change in $\arg\max_i R_i(k)$ for E.R. random and W.S. small world networks (the latter with mean degree 4), as a function of size n and the (re-)wiring probability (p for E.R. and β for W.S.). By construction, all the nodes in an E.R. random network are treated uniformly and randomly, resulting in a low probability that the network has a single distinct authority (unlike the B.A. networks considered in Example V.3). However, there is most often no significant difference between the nodes, and this lack of heterogeneity prevents p_{TV} to grow beyond ~ 0.2. On the contrary, the W.S. small-world networks [23] have both leader multiplicity and heterogeneity when the network size is large and the rewiring probability $\beta \sim 0.3$. For smaller or larger β, the network approaches a ring or an E.R. network, respectively, which we know have low p_{TV}. With $\beta \sim 0.3$, there is a sufficiently high probability of rewiring multiple nodes but there is a low probability of rewiring them all alike, resulting in multiplicity and heterogeneity of leaders. □

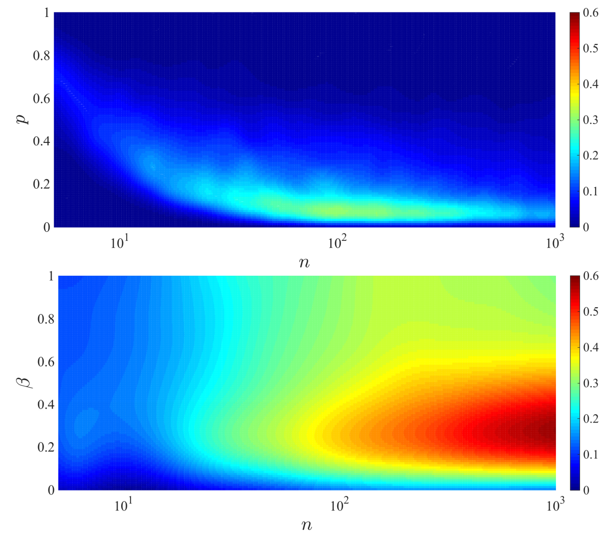

Fig. 4: The heat map of the probability of having at least one switch in $\arg\max_i R_i(k)$ as a function of network size n and, (top) wiring probability p for Erdös-Rényi random networks, (bottom) re-wiring probability β for Watts-Strogatz small-world networks.

Another class of networks for which the optimal actuator scheduling scheme is time-varying is those where the nodes with small (global) centrality have strong local connections within a subnetwork. The next result formalizes this statement by ensuring that increasing only the *local* weights of can turn them into the (globally) optimal control node(s).

Theorem VI.3. ***(Empowerment of subnetworks).*** *Given a network of n nodes with adjacency matrix $A_0 \in \mathbb{R}^{n\times n}$, let $E \in \mathbb{R}^{n\times n}$ be a symmetric nonnegative matrix of the form*

$$E = \begin{bmatrix} \overbrace{0}^{n-n_1} & \overbrace{0}^{n_1} \\ 0 & \star \end{bmatrix} \begin{matrix} \}^{n-n_1} \\ \}^{n_1} \end{matrix},$$

corresponding to a subnetwork involving the last $n_1 < n$ nodes (this is without loss of generality, since nodes can be renumbered). Let $i^* \in \{n - n_1 + 1, \ldots, n\}$ be the most *central node in E and consider the network described by $A = A_0 + \alpha E$, with $\alpha > 0$. Then, there exists $\overline{\alpha} > 0$ such that for $\alpha > \overline{\alpha}$, $R_{i^*}(k) > R_i(k)$, for $i \in \{1, \dots, n_1\}$ and $k \geq 1$.*

The significance of Theorem VI.3 is twofold. First, it ensures that locally central nodes can overcome the (globally) central ones in terms of $2k$-communicability, provided that their local subnetwork becomes sufficiently strong (i.e., as $\alpha > \overline{\alpha}$). This might be at first counter-intuitive for large k, as in this case the value of $2k$-communicability tends towards the global node centralities. Second, it suggests that for some $0 < \alpha < \overline{\alpha}$, the $2k$-communicabilities of nodes i^* and 1 are comparable, potentially leading to a time-varying $\arg\max_i R_i(k)$.

VII. Conclusions and Future Work

We have studied the optimality of time-varying actuator scheduling for maximizing the average controllability of single-input discrete-time linear networks. We have introduced the concept of $2k$-communicability and established that the trace of the controllability Gramian over some time horizon K is maximized by applying the input to the node with the largest $2k$-communicability at each time $K-1-k$. We have also identified several classes of networks for which time-invariant and time-varying actuator scheduling is beneficial. Our main conclusion is that, the more balanced (in terms of node centralities) *and* heterogeneous a network is, the more it will benefit from time-varying actuator scheduling. Numerous questions remain open for future work. Among these, we highlight the analysis of other controllability metrics beyond the trace of the Gramian, the generalization of our results to directed graphs, the development of tighter bounds on the number of optimal control node switches, the study of multiple-input networks and dynamic topologies.

Acknowledgments

This research was partially supported by NSF Award ECCS-1307176 and ONR award N00014-14-1-0816.

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Appendix I
$2k$-Communicabilities of Simple Networks

Proposition I.1. ***($2k$-communicabilities of line networks).*** *Consider a line network of n nodes with uniform link weights a (and no self-loops). For $i \in \{1, \dots, n\}$ and $k \in \mathbb{N}$,*

$$R_i(k) = a^{2k} \sum_{p \in \mathcal{I}} \left[\binom{2k}{k + p(n+1)} - \binom{2k}{k + p(n+1) - i} \right], \quad (6)$$

where $\mathcal{I} = \{-\lceil \frac{k}{n+1} \rceil, \dots, \lceil \frac{k}{n+1} \rceil\}$ and $\binom{n}{k} \triangleq 0$ if $k \notin \{0, \dots, n\}$. In particular, if $i \leq \lceil \frac{n}{2} \rceil$ and $k \leq \lceil \frac{n}{2} \rceil - 1$,

$$R_i(k) = a^{2k} \left[\binom{2k}{k} - \binom{2k}{k-i} \right]. \quad (7)$$

Proposition I.2. ***($2k$-communicabilities of ring networks).*** *Consider a ring network with n nodes and uniform link weights a (with no self-loops). For $i \in \{1, \dots, n\}$ and $k \in \mathbb{N}$,*

$$R_i(k) = \frac{(2a)^{2k}}{n} \left[1 + 2 \sum_{j=1}^{\lceil \frac{n}{2} \rceil - 1} \cos^{2k} \left(\frac{2j\pi}{n} \right) + \delta_n^E \right], \quad (8)$$

where δ_n^E equals one if n is even and zero otherwise. □

Proposition I.3. ***(2-communicability of star networks).*** *Consider a star network given by*

$$A = \begin{bmatrix} l_c & a^T \\ a & l_p I_{n-1} \end{bmatrix}, \quad (9)$$

where $a \in \mathbb{R}^{n-1}$ contains the link weights between the center node and peripheral nodes. A has three distinct eigenvalues and, for $i \in \{2, \dots, n\}$, $\frac{v_{11}}{v_{i1}} = \frac{l_c - l_p + \sqrt{(l_c - l_p)^2 + 4\|a\|^2}}{2a_{i-1}}$, and

$$R_1(1) - R_i(1) = l_c^2 - l_p^2 + \|a\|^2 - a_{i-1}^2. \quad (10)$$