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Fragility and Controllability Tradeoff in Complex Networks

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Abstract— **Mathematical theories and empirical evidence suggest that several complex natural and man-made systems are fragile: as their size increases, arbitrarily small and localized alterations of the system parameters may trigger system-wide failures. Examples are abundant, from perturbation of the population densities leading to extinction of species in ecological networks [1], to structural changes in metabolic networks preventing reactions [2], cascading failures in power networks [3], and the onset of epileptic seizures following alterations of structural connectivity among populations of neurons [4]. While fragility of these systems has long been recognized [5], convincing theories of why natural evolution or technological advance has failed, or avoided, to enhance robustness in complex systems are still lacking. In this paper we propose a mechanistic explanation of this phenomenon. We show that a fundamental tradeoff exists between fragility of a complex network and its controllability degree, that is, the control energy needed to drive the network state to a desirable state. We provide analytical and numerical evidence that easily controllable networks are fragile, suggesting that natural and man-made systems can either be resilient to parameters perturbation or efficient to adapt their state in response to external excitations and controls.**

I. Introduction

Across diverse scientific disciplines and application domains, complex systems are commonly represented as dynamic networks, where the interaction pattern among different parts is itself complex and may evolve along with the system dynamics. With this formalism, nodes and edges correspond, for instance, to populations of neurons and their functional relations in neural networks, or to different species and their trophic interactions in ecological networks, or to generators, loads and connection lines in power networks. Nodes sets are typically large; interconnections sparse and heterogeneous. Despite being able to accomplish a rich set of dynamic functionalities through different nodal and interconnection dynamics, many complex networks exhibit fragile behaviors against relatively small parameter variations. This is the case in ecological systems, where fragility affects the chance that species can coexist at a stable equilibrium, the variability of population densities over time, and the persistence of community composition [5]. In neuronal networks, fragility implies that small variations in certain synaptic weights can suddenly destabilize the network and cause seizures [4]. Small edge weight changes are also thought responsible for increases in ocean acidity [6], cascading failures in power systems [7], and traffic congestions [8].

Fragility of complex networks stands out as a negative feature, which, surprisingly, neither natural evolution nor engineering development have been able to remedy. Research in network science and graph optimization focuses primarily on static network models and diagnostics, e.g., see [9], [10], [11], [12], [13], and falls short in explaining network fragility. Only fewer and more recent work addresses dynamic network features, such as stability, fragility, and controllability [14], [15], [16], [17], [18]. Yet, to the best of our knowledge, a detailed link between fragility and controllability in networks has not been established yet. In this article we leverage network- and control-theoretic tools to form a mathematical explanation of why several natural and man-made networks are fragile. In particular, we show that a fundamental tradeoff exists between the fragility of a network and its controllability degree from exogenous inputs, and that certain systems may sacrifice their robustness in favor of an increased controllability degree.

Several definitions of fragility and controllability of a network have been proposed over the years and in different contexts. In this work, fragility measures the sensitivity of a network to variations of the edge weights. In particular, we quantify fragility of a stable network by measuring the norm of the smallest change in the network weights rendering the network unstable. To quantify the controllability degree of a system we use the control-theoretic notion of controllability Gramian. The controllability Gramian describes how signals propagate across a network, and its eigenvalues can be used to quantify the minimum control energy needed to steer the network state between different values. Optimized networks feature low fragility and high controllability, so as to remain stable against perturbations, yet allow for efficient manipulation from legitimate controls. Unfortunately, we show that these properties cannot be optimized simultaneously.

The contributions of this work are as follows. First, we derive an inequality involving the controllability and fragility degrees of a network – as measured, respectively, by the smallest eigenvalue of the Gramian and by the norm of the smallest perturbation rendering the network unstable – and the ratio of the number of control nodes to the total number of nodes (Theorem 3.1). In particular, this inequality and its refined version for symmetric networks (Corollary 3.2) show that the controllability degree of a network decreases linearly when the number of control nodes decreases and/or the network becomes less fragile. Although our inequalities provide a qualitative characterization of the fundamental

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tradeoff between controllability and fragility in networks, we also show (Remark 1) that tighter exponential bounds can be derived at the expenses of a more involved notation. Second, we quantify how the spectral and geometric properties of the network differentially determine controllability and fragility (Theorem 3.3). Specifically, we show that (see (20)) fragility depends upon the non-normality degree of the network, as measured by the condition number of the network eigenvectors matrix, and the stability margin of the network matrix, that is, the distance between the eigenvalues of the network matrix and the right-half complex plane. Further, the ratio of the condition number to the stability margin of the network constitute an upper bound for the smallest eigenvalue of the Gramian. This implies that (i) highly controllable networks must be fragile, (ii) normal[1] and highly-stable networks are robust but poorly controllable (as also highlighted in previous results, e.g., see [15]) (iii) less stable networks or highly non-normal networks are potentially more controllable but more fragile. Finally, we validate our results with numerical examples on a class of competitive predator-prey networks.

The rest of the paper is organized as follows. Section II contains the problem setup and the necessary preliminary notions. Section III presents our technical results showing that controllability and fragility are competing features in complex networks. Finally, Section IV contains our examples and numerical studies, and Section V concludes the paper.

II. Problem setup and preliminary notions

Consider a network represented by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively. Let $A = [a_{ij}]$ be the weighted adjacency matrix of $\mathcal{G}$, where $a_{ij} \in \mathbb{R} \setminus \{0\}$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. We assume that a subset of n_c nodes (drivers) can be controlled independently from one another and, to simplify the notation, we let the drivers be the first n_c nodes. The network dynamics are described by the following linear, continuous-time, and time-invariant model:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is the time-dependent vector of the nodes states, $e_i \in \mathbb{R}^n$ is the i-th canonical vector, $B = [e_1, \dots, e_{n_\text{c}}]$ is the input matrix, and $u : \mathbb{R} \to \mathbb{R}^{n_\text{c}}$ is the time-dependent vector containing the inputs injected into the driver nodes. We assume that the network matrix A is Hurwitz stable [19].

To quantify the controllability properties of the network (1) we resort to the controllability Gramian, which, for every control horizon $t_\text{f} > 0$, is defined as

$$W_{t_\text{f}} = \int_0^{t_f} e^{At} BB^\mathsf{T} e^{A^\mathsf{T} t} dt. \tag{2}$$

The controllability Gramian W_{t_f} is positive definite if and only if (1) is controllable, and positive semi-definite otherwise [20]. Further, the eigenvalues of W_{t_f} quantify the energy needed to control the state of the network (1) between any two states. For instance, if $x(0) = 0$, the minimum input energy required to control the network state to $x(t_\text{f}) = x_\text{f}$ is $x_\text{f}^\mathsf{T} W_{t_\text{f}}^{-1} x_\text{f}$. Thus, the larger the eigenvalues of the Gramian, the more controllable the network from the driver nodes [15].

The controllability Gramian can be computed in different ways. For instance, when the control horizon satisfies $t_\text{f} = \infty$, the controllability Gramian $W = W_\infty$ is the unique solution to the following Lyapunov equation:

$$AW + WA^\mathsf{T} = -BB^\mathsf{T}. \tag{3}$$

Equivalently [21], W can be computed explicitly as

$$W = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} (-BB^\mathsf{T}) (zI + A^\mathsf{T})^{-1} dz,$$

where Γ is any curve of the complex plane that encloses all eigenvalues of A, and i is the imaginary unit. By letting Γ be the semi-circle enclosing the left-half complex plane,

$$W = \frac{-1}{2\pi} \int_{-\infty}^{+\infty} (\omega \text{i} I - A)^{-1} BB^\mathsf{T} (\omega \text{i} I + A^\mathsf{T})^{-1} d\omega. \tag{4}$$

The expression (4) will be key in the derivation of our results.

We now introduce the concept of network fragility, which measures the ability of a network to maintain a stable behavior against perturbations of its weights. Specifically, we define the stability radius of the network (1) as[2]

$$r(A) = \min\{\|\Delta\| : A + \Delta \in \mathbb{C}^{n \times n} \text{ is not Hurwitz stable}\}.$$

When the stability radius $r(A)$ is small, then the network is fragile, because small changes in the network weights can induce unstable dynamics. Conversely, when $r(A)$ is large, the network maintains a stable behavior even after large perturbation of its weights, and is therefore robust. We will use the following equivalent characterization of $r(A)$ [23]:

$$r(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(\omega i I - A) = \frac{1}{\max_{\omega \in \mathbb{R}} \|(\omega i I - A)^{-1}\|}. \tag{5}$$

Clearly, $r(A) \leq \sigma_{\min}(A)$, the smallest singular value of A.

III. Fragility and controllability tradeoff in complex networks

In this section we derive inequalities to characterize a tradeoff between the controllability degree of a network and its fragility to perturbations. In particular, we show that controllability and fragility are directly related, so that networks that are easy to control tend to be fragile and non-fragile networks are difficult to control, and quantify that controllability and fragility are independently influenced by the algebraic and geometric structure of the network. Besides their theoretical value, our result constitute a first mathematical explanation of why several highly-optimized natural and technological networks are fragile (see Section IV).

Let $\lambda_{\min}(A)$, $\bar{\lambda}(A)$, and $\lambda_{\max}(A)$ denote the smallest, mean, and largest eigenvalue to the matrix A.

[1]A network is normal if its (real) matrix satisfies $AA^\mathsf{T} = A^\mathsf{T} A$ [19].

[2]We consider perturbations Δ with no particular structure. Thus, the matrix $A + \Delta$ may have a different sparsity pattern than A. See [22] for a similar notions of network observability radius with structured perturbations.

Theorem 3.1: ***(Controllability vs fragility)*** For the network (1) and for every $\alpha \in (0,1)$ it holds:

$$\lambda_{\min}(W) \le \bar{\lambda}(W) \le \frac{n_c}{n}\left(\frac{1}{2\alpha} + \frac{1}{\pi}\frac{\|A - A^{\mathsf{T}}\|}{(1-\alpha^2)}\frac{1}{r(A)}\right)\frac{1}{r(A)}. \tag{6}$$

Proof: Notice that

$$\begin{aligned}\bar{\lambda}(W) &\le \frac{1}{n}\lambda_{\max}\left(\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt\right)\mathrm{Tr}(BB^{\mathsf{T}}) \\ &= \frac{n_c}{n}\left\|\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt\right\|. \end{aligned}\tag{7}$$

Further, from (4) we obtain:

$$\begin{aligned}&\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt = \qquad (8)\\ &\quad = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\omega \mathrm{i}I - A^{\mathsf{T}})^{-1}(\omega\mathrm{i}I + A)^{-1}d\omega \\ &\quad = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}[(\omega\mathrm{i}I + A)(\omega\mathrm{i}I - A^{\mathsf{T}})]^{-1}d\omega \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\omega^2 I + AA^{\mathsf{T}} + \omega\mathrm{i}(A^{\mathsf{T}} - A)]^{-1}d\omega. \qquad (9)\end{aligned}$$

We now determine the values of ω satisfying

$$\omega^2 I + \omega\mathrm{i}(A^{\mathsf{T}} - A) \ge \alpha^2\omega^2 I \tag{10}$$

or, equivalently,

$$(1-\alpha^2)\omega^2 I + \mathrm{i}(A^{\mathsf{T}} - A)\omega \ge 0. \tag{11}$$

Observe that $A^{\mathsf{T}} - A$ is skew symmetric, and that $i(A^{\mathsf{T}} - A)$ is a Hermitian matrix [19]. It follows that the eigenvalues of $i(A^{\mathsf{T}} - A)$ are real and symmetric with respect to the origin. Namely, if μ is an eigenvalue of $i(A^{\mathsf{T}} - A)$, so is $-\mu$. Further, $i(A^{\mathsf{T}} - A)$ admits an orthogonal basis of eigenvectors, which implies that the maximum and the minimum eigenvalues of $i(A^{\mathsf{T}} - A)$ are $\|A^{\mathsf{T}} - A\|$ and $-\|A^{\mathsf{T}} - A\|$, respectively. This reasoning allows us to conclude that (11) holds if and only if

$$|\omega| \ge \bar{\omega} = \frac{\|A^{\mathsf{T}} - A\|}{1-\alpha^2}. \tag{12}$$

We now rewrite the integral (9) as

$$\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt = I_1 + I_2,$$

where

$$\begin{aligned}I_1 &= \frac{1}{2\pi}\int_{-\bar{\omega}}^{\bar{\omega}}[\omega^2 I + AA^{\mathsf{T}} + \omega\mathrm{i}(A^{\mathsf{T}} - A)]^{-1}d\omega, \\ I_2 &= \frac{1}{2\pi}\int_{-\infty}^{-\bar{\omega}}[\omega^2 I + AA^{\mathsf{T}} + \omega\mathrm{i}(A^{\mathsf{T}} - A)]^{-1}d\omega \\ &\quad + \frac{1}{2\pi}\int_{\bar{\omega}}^{+\infty}[\omega^2 I + AA^{\mathsf{T}} + \omega\mathrm{i}(A^{\mathsf{T}} - A)]^{-1}d\omega.\end{aligned}$$

From (9) and (12) it follows that

$$\begin{aligned}I_1 &\le \frac{\bar{\omega}}{\pi}\max_{\omega\in[0,\bar{\omega}]}\|(\omega\mathrm{i}I - A^{\mathsf{T}})^{-1}\|^2 \\ &\le \frac{\bar{\omega}}{\pi}\max_{\omega\in\mathbb{R}}\|(\omega\mathrm{i}I - A^{\mathsf{T}})^{-1}\|^2 \\ &= \frac{\bar{\omega}}{\pi}\frac{1}{r(A^{\mathsf{T}})^2} = \frac{\bar{\omega}}{\pi}\frac{1}{r(A)^2} \\ &= \frac{1}{\pi}\frac{\|A - A^{\mathsf{T}}\|}{1-\alpha^2}\frac{1}{r(A)^2}.\end{aligned}$$

Similarly, from (10) and (12) it follows that

$$\begin{aligned}I_2 &\le \frac{1}{2\pi}\int_{-\infty}^{-\bar{\omega}}[\alpha^2\omega^2 I + AA^{\mathsf{T}}]^{-1}d\omega \\ &\quad + \frac{1}{2\pi}\int_{\bar{\omega}}^{+\infty}[\alpha^2\omega^2 I + AA^{\mathsf{T}}]^{-1}d\omega \\ &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\alpha^2\omega^2 I + AA^{\mathsf{T}}]^{-1}d\omega.\end{aligned}$$

Because AA^{T} is symmetric, we have

$$AA^{\mathsf{T}} = U^{\mathsf{H}}\,\mathrm{diag}(\sigma_i(A)^2)\,U, \tag{13}$$

where U is a unitary matrix, $\sigma_i(A)$ are the singular values of A, and $\mathrm{diag}(s_i)$ is the diagonal matrix of the elements s_i. Then, the integral I_2 can be upper bounded as

$$\begin{aligned}I_2 &\le \frac{1}{2\pi}U^{\mathsf{H}}\int_{-\infty}^{+\infty}\mathrm{diag}\left(\frac{1}{\alpha^2\omega^2 + \sigma_i(A)^2}\right)d\omega\, U \\ &= \frac{1}{2\pi}U^{\mathsf{H}}\,\mathrm{diag}\left(\left[\frac{1}{\alpha\sigma_i(A)}\arctan\left(\frac{\alpha}{\sigma_i(A)}\omega\right)\right]_{-\infty}^{\infty}\right)U \\ &= \frac{1}{2\pi}U^{\mathsf{H}}\,\mathrm{diag}\left(\frac{\pi}{\alpha\sigma_i(A)}\right)U = \frac{1}{2\alpha}U^{\mathsf{H}}\,\mathrm{diag}\left(\frac{1}{\sigma_i(A)}\right)U.\end{aligned}$$

Consequently, we have that

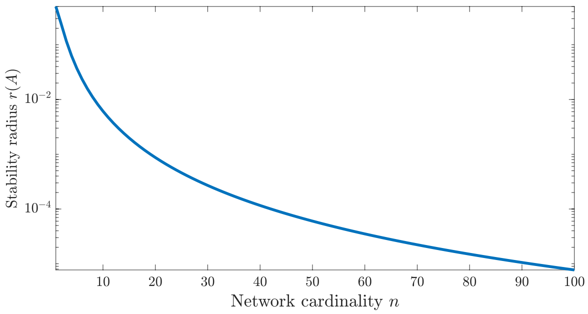
$$\|I_2\| \le \frac{1}{2\alpha\sigma_{\min}(A)}$$

where $\sigma_{\min}(A)$ is the smallest singular value of A. To conclude, (5) implies that $r(A) \le \sigma_{\min}(A)$, which leads to

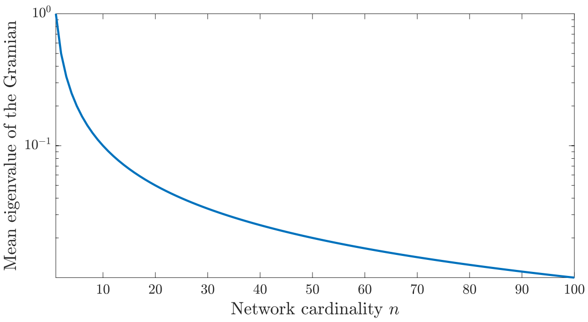
$$\left\|\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt\right\| \le \frac{1}{\pi}\frac{\|A - A^T\|}{1-\alpha^2}\frac{1}{r(A)^2} + \frac{1}{2\alpha}\frac{1}{r(A)}. \tag{14}$$

■

Theorem 3.1 provides a family of inequalities that reveal a number of fundamental tradeoffs between the controllability degree of a network, its fragility, and the number of driver nodes. First, the fewer the driver nodes, the smaller the Gramian eigenvalue $\lambda_{\min}(W)$ and, consequently, the larger the energy needed to control the network to certain states. Second, the larger the stability radius $r(A)$, the smaller the Gramian eigenvalue $\lambda_{\min}(W)$, thus proving that robust networks cannot be easy to control. Third, when the network dimension n grows, the number of driver nodes n_c remains constant, and $\|A - A^{\mathsf{T}}\|/r(A)$ depends sub-linearly on n, then the product $\lambda_{\min}(W)r(A)$ decreases, proving a decrease of controllability (small $\lambda_{\min}(W)$) or a loss of robustness



(a)



(b)

Fig. 1. Fig. (a) shows the fragility degree of the network A_1 in Example 1. As predicted by our results, because the controllability degree of A_1 is independent of the network cardinality n, the network becomes more fragile as n the network cardinality increases. Fig. (b) shows the mean eigenvalue of the controllability Gramian of the network A_2 in Example 1. As predicted by our results, because the fragility of the network satisfies $r(A_2) = 0.5$ independently of n, the network controllability degree as measured by the mean eigenvalue of the Gramian decreases as the cardinality n increases.

(small $r(A)$). Theorem 3.1 can be refined in different ways. For instance, by letting $\alpha = 0.5$ we obtain

$$\lambda_{\text{min}}(W) \leq \bar{\lambda}(W) \leq \frac{n_{\text{c}}}{n}\left(1 + \frac{4\|A - A^{\mathsf{T}}\|}{3\pi}\frac{1}{r(A)}\right)\frac{1}{r(A)}. \tag{15}$$

In fact, an optimal bound can be computed by minimizing the right-hand side of (15) over the parameter α. Moreover, the result further simplifies when the matrix A is symmetric.

Corollary 3.2: ***(Controllability vs fragility in symmetric networks)*** For the network (1), if $A = A^{\mathsf{T}}$, then

$$\lambda_{\text{min}}(W) \leq \bar{\lambda}(W) \leq \frac{n_{\text{c}}}{n}\frac{1}{2r(A)}. \tag{16}$$

Proof: Because $A = A^{\mathsf{T}}$, from (6) we obtain

$$\lambda_{\text{min}}(W) \leq \bar{\lambda}(W) \leq \frac{n_{\text{c}}}{n}\frac{1}{2\alpha}\frac{1}{r(A)}.$$

The statement follows by selecting $\alpha = 1$. ■

Example 1: ***(Example of fragile and easy to control networks)*** To illustrate our results, consider a network with n nodes, $n_{\text{c}} = 1$, and adjacency matrix $A_1 = [a_{ij}]$, where

$$a_{ij} = \begin{cases} -1/2, & \text{if } i = j = 1, \\ -1, & \text{if } j = i + 1, \\ 1, & \text{if } j = i - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

It can be verified that the controllability Gramian equals the n-dimensional identity matrix. Thus, the network A_1 is easy to control, in the sense that its control energy is independent of the network cardinality n. See also [24]. Yet, as illustrated in Fig. 1(a), the fragility of the network increases with the network cardinality, thus showing the discussed tradeoff.

Consider now the a network with n nodes, $n_{\text{c}} = 1$, and adjacency matrix $A_2 = [a_{ij}]$, where

$$a_{ij} = \begin{cases} -1/2, & \text{if } i = j, \\ -1, & \text{if } j = i + 1, \\ 1, & \text{if } j = i - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

It can be verified that A_2 is a normal matrix, and that $r(A_2) = 0.5$ independently of the network cardinality. Thus, the network A_2 is robust to perturbation. Yet, as illustrated in Fig. 1(b), the mean eigenvalue of the Gramian decreases with the network cardinality, showing that the network becomes more difficult to control as the cardinality increases. □

To reveal the network properties that determine the controllability and fragility degrees, we next restrict our analysis to diagonalizable networks. That is, we now assume that the matrix A can be written as $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix containing the eigenvalues of A. Let $\kappa(V) = \sigma_{\text{max}}(V)/\sigma_{\text{min}}(V)$ be the condition number of V, and define the stability margin of A as

$$s(A) = -\max_{i \in \{1,\dots,n\}} \Re(\lambda_i(A)),$$

where $\Re(\lambda_i(A))$ denotes the real part of $\lambda_i(A)$. Notice that $s(A) > 0$ when A is stable, and that, generally, $s(A) \neq r(A)$.

Theorem 3.3: ***(Properties that determine controllability and fragility)*** For the network (1), if A is diagonalizable as $A = V\Lambda V^{-1}$, then

$$\lambda_{\text{min}}(W) \leq \bar{\lambda}(W) \leq \frac{n_{\text{c}}}{n}\frac{\kappa^2(V)}{2s(A)}. \tag{19}$$

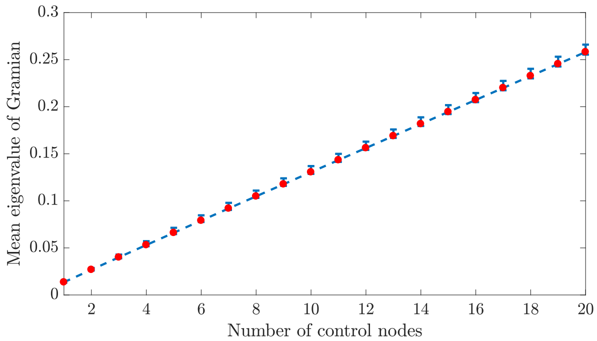
Proof: Notice that

$$\begin{aligned} &\sigma_{\text{max}}\left(\int_0^\infty e^{A^{\mathsf{T}}t}e^{At}dt\right) \\ &\quad = \sigma_{\text{max}}\left(\int_0^\infty V^{-\mathsf{H}}e^{\Lambda^{\mathsf{H}}t}V^{\mathsf{H}}Ve^{\Lambda t}V^{-1}dt\right) \\ &\quad \leq \sigma_{\text{max}}^2(V)\sigma_{\text{max}}^2(V^{-1})\sigma_{\text{max}}\left(\int_0^\infty e^{\Lambda^{\mathsf{H}}t}e^{\Lambda t}dt\right) \\ &\quad = \kappa^2(V)\max_i \frac{1}{-2\Re(\lambda_i(A))} = \frac{\kappa^2(V)}{2s(A)}. \end{aligned}$$

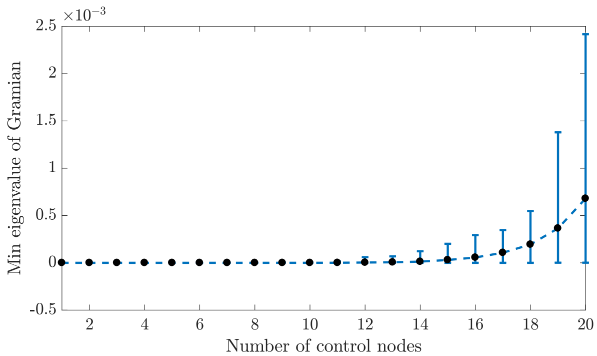
The claimed statement follows from equation (7). ■

Theorem 3.3 shows that the controllability degree is inversely proportional to the stability margin. In particular, when $s(A)$ grows and n, n_{c}, and $k(V)$ remain bounded, the eigenvalue $\lambda_{\text{min}}(W)$ must decrease. By applying the Bauer-Fike Theorem [25], it can additionally be shown that

$$\frac{s(A)}{\kappa(V)} \leq r(A) \leq s(A). \tag{20}$$



(a)



(b)

Fig. 2. This figure shows the mean and smallest eigenvalue of the controllability Gramian of random geometric networks with 20 nodes and connectivity radius 0.6. The network matrix is a weighted Laplacian, where each edge weight equals the inverse of the distance between its end nodes and where the diagonal elements equal the negative sum of the off-diagonal entries minus 0.1. Thus, these network are Hurwitz stable. Figs. (a) and (b) show the mean and smallest eigenvalues of the Gramian, respectively, averaged over 100 network instances. It can be seen that the mean eigenvalue depends linearly on the number of control nodes, while the smallest eigenvalue depends exponentially on the number of control nodes.

Together, equations (19) and (20) show that, when $k(V)$ is bounded, less fragile networks are also less controllable, and the stability margin $s(A)$ is the cause of this tradeoff. While the role of $k(V)$ on controllability and fragility cannot be inferred from (19) and (20), these equations suggest that increasing $k(V)$, also known as "non-normality degree" [26], may lead to larger controllability and fragility degrees.

The result in Theorem 3.3 simplifies for normal networks ($\kappa(V) = 1$) [19], and it becomes

$$\lambda_{\text{min}}(W) \leq \bar{\lambda}(W) \leq \frac{n_{\text{c}}}{n} \frac{1}{2s(A)}.$$

To conclude this section, in the following remark we discuss the tightness of the above inequalities.

Remark 1: ***(Linear and exponential eigenvalues decay)*** The inequalities (6), (16), and (19) reveal a tradeoff between the controllability degree of a network, its fragility to parameters perturbations, and the number of driver nodes. However, while these inequalities suggest that the smallest eigenvalue of the Gramian depends linearly on the ratio n_{c}/n, this relation may be exponential as for the case of discrete-time network systems [15]. To see this, recall from [27] that the eigenvalues of the Gramian satisfy the inequality

$$\lambda_{n_{\text{c}}k+1}(W) \leq \kappa^2(V)\rho^k \lambda_{\text{max}}(W), \text{ with } n_{\text{c}}k + 1 \leq n,$$

where $\rho < 1$ depends only on the eigenvalues of A. Then,

$$\lambda_{\text{min}}(W) \leq \lambda_n(W) \leq \kappa^2(V)\rho^{\frac{n-1}{n_{\text{c}}}} \lambda_{\text{max}}(W).$$

By the same argument as in the proof of Theorem 3.3, we obtain $\lambda_{\text{max}}(W) \leq \kappa^2(V)/(2s(A))$. We conclude that

$$\lambda_{\text{min}}(W) \leq \frac{\kappa^4(V)}{2s(A)} \rho^{\frac{(n-1)}{n_{\text{c}}}},$$

which proves that the smallest eigenvalue of the Gramian depends exponentially on the ratio n_{c}/n (assuming that ρ remains upper bounded by a constant $\bar{\rho} < 1$ as n increases).

Although our inequalities are loose for $\lambda_{\text{min}}(W)$, in Fig. 2 we provide numerical evidence that that our bounds tightly capture the behavior of the mean eigenvalue $\bar{\lambda}(W)$. □

IV. Controllability and Fragility in Competitive Predator-Prey Networks

To illustrate our results we focus on networks arising from the linearization of Lotka-Volterra predator-prey systems, which describe the dynamic interaction of various competing and cooperating species in a restricted environment; e.g., see [28]. For a system with n species, the population density x_i of the i-th specie is described by the differential equation

$$\dot{x}_i = x_i \left(g_i + \sum_{j=1}^{n} a_{ij} x_j \right), \tag{21}$$

where $g_i \in \mathbb{R}$ is the growth coefficient of specie i, and $a_{ij} \in \mathbb{R}$ specifies whether specie i benefits ($a_{ij} > 0$) or suffers ($a_{ij} < 0$) from the presence of specie j in the community [26, Chapter XI]. For our numerical study we assume that $g_i > 0$, $a_{ii} < 0$, and that $a_{ij} = -a_{ji}$ for all indices $i \neq j$ (competitive interaction among species). Further, we assume that the species are at equilibrium x^{eq}, which can be obtained by solving the equation $g = -Ax^{\text{eq}}$, where g is the vector of growth rates and $A = [a_{ij}]$. In a neighborhood of x_{eq}, the network dynamics are captured by the Jacobian matrix of (21), which can be written in matrix form as $J = \text{diag}(x_{\text{eq}})A$.

To characterize how controllability and fragility are related in predator-prey networks, first we randomly generate adjacency matrices A reflecting competitive interconnection among n species and equilibrium vectors x_{eq}, and compute the associated Jacobian matrices $J = \text{diag}(x_{\text{eq}})A$. Then, for all networks corresponding to stable equilibrium configurations, we evaluate and plot the mean eigenvalue of the network controllability Gramian versus the fragility index $r(J)$ for the best choice of n_{c} control nodes. The results of our numerical study are reported in Fig. 3, where we see that controllability and robustness are indeed inversely related.

Lastly, for the class of predator-prey networks described above, in Fig. 4 we compare the bound obtained in Theorem 3.1, particularly (15), with the inequality in Theorem 3.3.

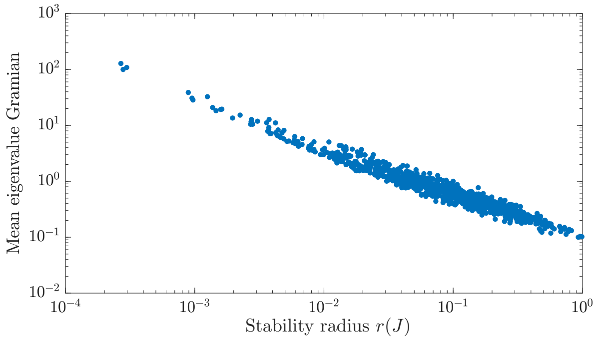


Fig. 3. This figure plots in logarithmic scales the mean eigenvalue of the controllability Gramian versus the fragility degree $r(J)$ of 1000 randomly generated predator-prey networks of dimension 20 (see Section IV). The controllability Gramian is obtained for the set of 5 control nodes that maximizes its mean eigenvalue. As we show in Theorems 3.1 and 3.3, controllability and robustness are inversely related network properties.

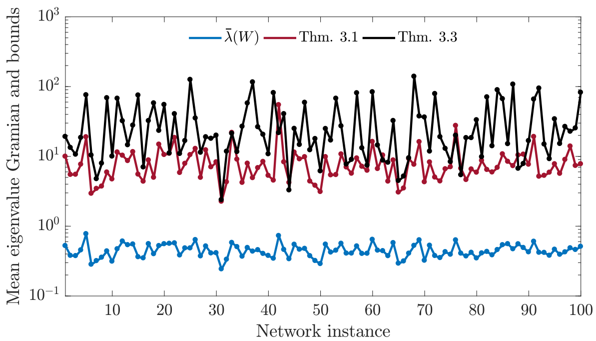


Fig. 4. This figure compares in a logarithmic scale the mean eigenvalue of the Gramian (blue) with the bounds in (15) (red) and Theorem 3.3 (black) for 100 randomly generated ecological networks. For this class of networks, Theorem 3.1 seems to provide a tighter bound than Theorem 3.3.

V. Conclusion

In this paper we study controllability and fragility of complex networks. Controllability measures the energetic effort needed to steer the network state between desirable configurations, and is quantified by the eigenvalues of the network controllability Gramian. Fragility, instead, measures the ability of a network to maintain stability against perturbations of its edge weights, and is quantified by the norm of the smallest perturbation rendering the network matrix unstable. We provide analytical and numerical evidence that controllability and robustness are inversely related, effectively showing that robust networks are difficult to control. Further, we characterize algebraic and geometric properties of the network matrix contributing to controllability and fragility. In particular, we show that fragility depends on the non-normality degree and the stability margin of the network matrix, and that their ratio constitutes an upper bound for the mean eigenvalue of the Gramian. Finally, we numerically investigate tightness of our bounds, and illustrate our theories through a class of competitive predator-prey networks.

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