# Learning Minimum-Energy Controls from Heterogeneous Data

Giacomo Baggio and Fabio Pasqualetti

***Abstract*— In this paper we study the problem of learning minimum-energy controls for linear systems from heterogeneous data. Specifically, we consider datasets comprising input, initial and final state measurements collected using experiments with different time horizons and arbitrary initial conditions. In this setting, we first establish a general representation of input and sampled state trajectories of the system based on the available data. Then, we leverage this data-based representation to derive closed-form data-driven expressions of minimum-energy controls for a wide range of control horizons. Further, we characterize the minimum number of data required to reconstruct the minimum-energy inputs, and discuss the numerical properties of our expressions. Finally, we investigate the effect of noise on our data-driven formulas, and, in the case of noise with known second-order statistics, we provide corrected expressions that converge asymptotically to the true optimal control inputs.**

## I. Introduction

The availability of large volumes of freely accessible data and the recent advances in machine learning and artificial intelligence are revolutionizing many areas of science and engineering. These include control and system theory, in which direct data-driven control design has recently been recognized as an appealing (and sometimes preferable) alternative to the classic model-based paradigm [1]–[6]. In particular, learning controls directly from data turns out to be beneficial when an accurate model of the system is difficult or expensive to obtain from first principles, or when system identification leads to significant errors or excessive computational costs in the reconstruction of the desired control.

Several direct data-driven control design approaches have been proposed and analyzed in the literature (see [7] for an overview of recent results). These differ in the class of dynamics, control objective, and data collection, and include, among others, (model-free) reinforcement learning [8], iterative learning control [9], adaptive control [10], and behavior- or subspace-based methods [1], [5], [11].

In this paper, we focus on learning the minimum-energy control input driving a linear system from an initial state to a desired target one. We show that this control input can be exactly reconstructed from data consisting of heterogeneous and, in certain cases, noisy measurements of system trajectories. In particular, we establish closed-form data-driven expressions of minimum-energy controls for noiseless and noisy data. Besides further supporting the intriguing idea that data-driven control represents a viable alternative to model-based control, our framework and results offer a different, attractive perspective on many problems in network analysis and control. In fact, (model-based) minimum-energy controls have been extensively employed for controlling, and characterizing the control performance of, large-scale networks governed by linear dynamics, e.g., see [12]–[14].

This material is based upon work supported in part by awards ARO 71603NSYIP, ARO W911NF-18-1-0213, and AFOSR FA9550-19-1-0235. Giacomo Baggio is with the Department of Information Engineering, University of Padova, Italy, e-mail: baggio@dei.unipd.it. Fabio Pasqualetti is with the Department of Mechanical Engineering, University of California at Riverside, e-mail: fabiopas@engr.ucr.edu.

**Related work.** The data-driven framework employed in this paper is similar to the one of [1], [3], [4], which can, in turn, be viewed as a state-space adaptation of the behavioral setting described in, e.g., [2], [11], [15]. These works exploit a data-based representation of the system in terms of data that typically consist of uninterrupted samples of a single, noiseless, and sufficiently long input-output trajectory. Here, instead, we consider data collected from system trajectories with possibly different time horizons and initial conditions. Further, under some assumptions on the noise model, we establish asymptotic results for case of data corrupted by noise. Finally, besides our earlier work [6], we are not aware of data-driven approaches tailored to minimum-energy controls.

**Contribution.** The contributions of this paper are threefold. First, we provide a data-based representation of sampled system trajectories based on data comprising input, initial and final state measurements collected via control experiments with different time horizons, arbitrary inputs and initial conditions. Second, based on these data, we establish two equivalent closed-form expressions of the minimum-energy control input to reach a desired target state. Differently from [6], our expressions can be used to compute minimum-energy controls for a wide range of control times, and, in particular, for times that are determined only by the experimental data and that can exceed the largest time horizon of the collected experiments. Further, we discuss the numerical properties of our data-driven expressions, and the minimum number of data required to correctly reconstruct the minimum-energy control inputs. Third and finally, in the case of data corrupted by noise with known second-order statistics, we propose corrected data-driven control expressions, and show that these converge to the true control inputs in the limit of infinite data.

**Organization.** The rest of the paper is organized as follows. In Section II, we illustrate the class of systems and data collection setting considered in this paper. In Section III, we establish a data-based parameterization of sampled system trajectories. In Section IV and V, we present and discuss data-driven expressions of minimum-energy controls for the case of noiseless and noisy data, respectively. Finally, Section VI contains some concluding remarks and future directions.

**Notation.** Given a matrix $A \in \mathbb{R}^{p\times q}$, we let $\mathrm{Ker}(A)$ and $A^\dagger$ denote the kernel and Moore–Penrose pseudoinverse of $A$, respectively. We let $0_{n,m}$ and $I_n$ denote the $n \times m$ zero matrix (we simply write $0_n$ if $m = n$) and $n \times n$ identity matrix, respectively. We will omit the subscripts when the dimensions are clear from the context. Further, we denote with $K_A$ the matrix whose columns form a basis of $\mathrm{Ker}(A)$.

## II. System dynamics and available data

Consider a discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and input of the system at time $t$, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the state and input matrices, respectively. Let $C_T = [B \;\; AB \;\; \cdots \;\; A^{T-1}B]$ denote the $T$-steps controllability matrix of the system (1). We assume that $A$ and $B$ are unknown, and that a set of control experiments with the system (1) has been conducted for control purposes. Each control experiment consists of (i) generating a $T$-steps input sequence $u_T = [u(T-1)^\mathsf{T}, \dots, u(0)^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{mT}$, and (ii) measuring the state of the system with input $u_T$ at time $t = 0$, namely $x(0)$, and at time $t = T$, namely,

$$x(T) = A^T x(0) + C_T u_T. \qquad (2)$$

We assume that the control experiments have been performed using $M$ distinct time horizons $T_i \in \mathbb{N}$, $i \in \{1, \dots, M\}$, and we divide the available data in sets $(U_i, X_{0,i}, X_i)$, $i \in \{1, \dots, M\}$, where the $i$-th set contains $N_i$ experiments, and $U_i \in \mathbb{R}^{mT_i\times N_i}$, $X_{0,i} \in \mathbb{R}^{n\times N_i}$, and $X_i \in \mathbb{R}^{n\times N_i}$ denote the matrices whose columns contain, respectively, the input sequences with horizon $T_i$, the initial states of the experiments, and the final state measurements recorded at time $T_i$. We let $\mathcal{D} = \{(U_i, X_{0,i}, X_i)\}_{i=1}^M$ denote the set of all available data.

We stress that, equivalently, $\mathcal{D}$ may comprise measurements that have (intermittently) been recorded from a sufficiently long experiment or from several short and independent ones (possibly performed using different initializations). The first scenario is quite standard for system identification [16] and behavior-based control [1], where data typically consist of a single system trajectory (the case of missing observations has been analyzed in a limited number of works, e.g., see [17]). The second experimental scenario has recently been considered in [6], [18], under the more restrictive assumption that the initial state is the same for all experiments.

## III. Data-based representation of sampled system trajectories

Consider a sequence of (possibly repeated) indices $k_1, \dots, k_\ell \in \{1, \dots, M\}$, and let $T = \sum_{i=1}^{\ell} T_{k_i}$. Further, let

$$x_{k_1,\dots,k_\ell} = \left[x(0)^\mathsf{T}, x(T_{k_1})^\mathsf{T}, x(T_{k_1}+T_{k_2})^\mathsf{T}, \dots, x(T)^\mathsf{T}\right]^\mathsf{T}$$

denote the state trajectory of (1) generated by the control input $u_T \in \mathbb{R}^{mT}$ and sampled at times $0, T_{k_1}, T_{k_1}+T_{k_2}, \dots, T$. For notational convenience, we write $x_{0:T}$ when $T_{k_i} = 1$ for all $i$. The next result provides a parameterization of all admissible pairs $(u_T, x_{k_1,\dots,k_\ell})$ in terms of the data $\mathcal{D}$.

*Theorem 3.1:* ***(Data-based representation of input and sampled state pairs)*** If $[X_{0,k_i}^\mathsf{T} \;\; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1, \dots, \ell\}$, then any pair $(u_T, x_{k_1,\dots,k_\ell})$ of input and sampled state trajectories of the system (1) satisfies

$$\begin{bmatrix} u_T \\ x_{k_1,\dots,k_\ell} \end{bmatrix} = \begin{bmatrix} G \\ H \end{bmatrix} \alpha, \quad \alpha \in \mathbb{R}^{q_{k_\ell}+\dots q_{k_1}+n}, \qquad (3)$$

where $q_{k_i} = \dim \mathrm{Ker}(X_{0,k_i})$, and

$$G = \begin{bmatrix} \tilde{U}_\ell & 0 & \cdots & 0 & 0_n \\ 0 & \tilde{U}_{\ell-1} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0_n \\ 0 & \cdots & 0 & \tilde{U}_1 & 0_n \end{bmatrix}, \qquad (4)$$

$$H = \begin{bmatrix} 0 & \cdots & 0 & 0 & I \\ 0 & \cdots & 0 & \tilde{X}_1 & Q_1 \\ \vdots & \iddots & \tilde{X}_2 & Q_2\tilde{X}_1 & Q_2Q_1 \\ 0 & \iddots & \vdots & \vdots & \vdots \\ \tilde{X}_\ell & \cdots & \prod_{i=0}^{\ell-3} Q_{\ell-i}\tilde{X}_2 & \prod_{i=0}^{\ell-2} Q_{\ell-i}\tilde{X}_1 & \prod_{i=0}^{\ell-1} Q_{\ell-i} \end{bmatrix}, \qquad (5)$$

with $\tilde{U}_i = U_{k_i} K_{X_{0,k_i}}$, $\tilde{X}_i = X_{k_i} K_{X_{0,k_i}}$, and $Q_i = X_{k_i} K_{U_{k_i}} (X_{0,k_i} K_{U_{k_i}})^\dagger$, for all $i \in \{1, \dots, \ell\}$.

*Proof:* Note that, since $[X_{0,k_i}^\mathsf{T} \;\; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1, \dots, \ell\}$, $\tilde{U}_i = U_{k_i} K_{X_{0,k_i}}$ is full row rank for all $i \in \{1, \dots, \ell\}$.[1] From (4), this implies that $G$ is full row rank, and, therefore, for every $T$-steps input sequence $u_T$ there exists a real vector $\alpha$ such that $u_T = G\alpha$. We next show that the sampled state $x_{k_1,\dots,k_\ell}$ corresponding to the input $u_T = G\alpha$ can be expressed as $H\alpha$, with $H$ as in (5). To this aim, let $C_{T_i}$ denote the $T_i$-steps controllability matrix of (1), and observe that, for all $j \in \{1, \dots, \ell\}$,

$$\begin{aligned} x(T_{k_1} + \cdots + T_{k_j}) = {} & A^{T_{k_1}+\cdots+T_{k_j}} x_0 + \\ & + A^{T_{k_2}+\cdots+T_{k_j}} C_{T_{k_1}} \tilde{U}_1 \alpha_1 + \cdots + C_{T_{k_j}} \tilde{U}_j \alpha_j, \end{aligned} \qquad (6)$$

where we partitioned $\alpha$ as $\alpha = [\alpha_\ell^\mathsf{T}, \alpha_{\ell-1}^\mathsf{T}, \dots, \alpha_1^\mathsf{T}, \alpha_0^\mathsf{T}]^\mathsf{T}$, with $\alpha_i \in \mathbb{R}^{q_{k_i}}$, and $\alpha_0 \in \mathbb{R}^n$. Set $\alpha_0 = x_0$. From

$$\begin{aligned} \tilde{X}_i = X_{k_i} K_{X_{0,k_i}} &= (A^{T_{k_i}} X_{0,k_i} + C_{T_{k_i}} U_{k_i}) K_{X_{0,k_i}} \\ &= C_{T_{k_i}} U_{k_i} K_{X_{0,k_i}} = C_{T_{k_i}} \tilde{U}_i, \end{aligned}$$

it follows that (6) can be rewritten as

$$\begin{aligned} x(T_{k_1} + \cdots + T_{k_j}) = {} & A^{T_{k_1}+\cdots+T_{k_j}} \alpha_0 + \\ & + A^{T_{k_2}+\cdots+T_{k_j}} \tilde{X}_1 \alpha_1 + \cdots + \tilde{X}_j \alpha_j. \end{aligned} \qquad (7)$$

Additionally, because $[X_{0,k_i}^\mathsf{T} \;\; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank, $X_{0,k_i} K_{U_{k_i}}$ is full row rank, and from

$$\begin{aligned} X_{k_i} K_{U_{k_i}} &= (A^{T_{k_i}} X_{0,k_i} + C_{T_{k_i}} U_{k_i}) K_{U_{k_i}} \\ &= A^{T_{k_i}} X_{0,k_i} K_{U_{k_i}}, \end{aligned}$$

it follows that

$$Q_i = X_{k_i} K_{U_{k_i}} (X_{0,k_i} K_{U_{k_i}})^\dagger = A^{T_{k_i}}. \qquad (8)$$

Finally, by substituting (8) into (7) and rewriting the latter in vector form, we obtain $x_{k_1,\dots,k_\ell} = H\alpha$, with $H$ as in (5). ■

The previous result states that any $T$-steps input sequence and corresponding state trajectory sampled at times $0, T_{k_1}, T_{k_1}+T_{k_2}, \dots, T$ of the system (1) can be written as a linear combination of the columns of a matrix that depends

[1] Indeed, since $[X_{0,k_i}^\mathsf{T} \;\; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank, for all $u \in \mathbb{R}^{mT}$ there exists $\gamma \in \mathrm{Ker}(X_{0,k_i})$ such that $[0 \;\; u^\mathsf{T}]^\mathsf{T} = [X_{0,k_i}^\mathsf{T} \;\; U_{k_i}^\mathsf{T}]^\mathsf{T}\gamma$, which implies that $U_{k_i} K_{X_{0,k_i}}$ must be of full row rank.

on the dataset $\mathcal{D}$ only. Intuitively, this sampled data-based representation is obtained by suitably "gluing" together the data-based representations of system trajectories of lengths $T_{k_1}, T_{k_2}, \ldots, T_{k_\ell}$. One of the advantages of our parameterization is that it provides a data-based description of a linear system that does not rely on the identification of the system matrices $A$ and $B$. Further, when the full state of the system is accessible, the data-based representation of Theorem 3.1 generalizes those employed in a number of recent works (e.g., [1], [2], [5]), which rely on measurements of a single, uninterrupted, and sufficiently long input-output trajectory.[2] To clarify the notation and implications of Theorem 3.1, we next illustrate our result by means of a simple example.

*Example 1:* ***(Illustration of Theorem 3.1)*** Consider the scalar system

$$x(t+1) = ax(t) + u(t), \quad a \in \mathbb{R}, \tag{9}$$

and assume that $M = 1$, $N_1 = 3$, $T_1 = 2$, that is, data have been generated from three control experiments performed using a single time horizon of length two. Further, consider the following dataset $\mathcal{D} = \{(U_1, X_{0,1}, X_1)\}$, where

$$U_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; X_{0,1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; X_1 = \begin{bmatrix} a^2 & 1 & a \end{bmatrix}.$$

Notice that $[X_{0,1}^\mathsf{T} \; U_1^\mathsf{T}]^\mathsf{T}$ has full row rank, and that

$$K_{U_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; K_{X_{0,1}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; Q_1 = a^2.$$

Thus, by choosing $\ell = 2$ and $k_1 = k_2 = 1$, by Theorem (3.1), any input $u_T$ and resulting state sampled at time 0, $T_1 = 2$, $T = 2T_1 = 4$, $x_{0,2,4}$, of (9) satisfy (3), where

$$G = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \; H = \left[\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & a & a^2 \\ \hline 1 & a & a^2 & a^3 & a^4 \end{array}\right].$$

We note, in particular, that to compute the matrices $G$ and $H$, we did not reconstruct the system parameter $a$. □

When the dataset $\mathcal{D}$ contains trajectories recorded using a unit-length time horizon[3], we have the following immediate corollary of Theorem 3.1, which provides a complete data-based parameterization of all input sequences and corresponding state trajectories of the system (1).

*Corollary 3.2:* ***(Complete data-based representation of input and state pairs)*** Assume that there exists an index $j \in \{1, \ldots, M\}$ such that $T_j = 1$. If $[X_{0,j}^\mathsf{T} \; U_j^\mathsf{T}]^\mathsf{T}$ is full row rank, then, for any $T \geq 1$, any pair of input $u_T$ and corresponding state trajectory $x_{0:T}$ of the system (1) satisfies

$$\begin{bmatrix} u_T \\ x_{0:T} \end{bmatrix} = \begin{bmatrix} G \\ H \end{bmatrix} \alpha, \quad \alpha \in \mathbb{R}^{Tq_j + n}, \tag{10}$$

where $G$ and $H$ are defined as in (4) and (5), respectively, with $\ell = T$ and $k_i = j$ for all $i \in \{1, \ldots, \ell\}$.

[2]A partial extension of this setting to multiple measured trajectories has been proposed in [4], [19], under the rather restrictive assumption that these trajectories align over a sufficiently long window at their intersection.

[3]We remark that a unit-length dataset can be constructed from measurements of a single trajectory by dividing the latter into unit-length segments.

## IV. Closed-form data-driven expressions of minimum-energy controls

### A. Problem formulation

For a control horizon $T \geq 1$ and desired initial and final states $x_0 \in \mathbb{R}^n$ and $x_\mathrm{f} \in \mathbb{R}^n$, respectively, the minimum-energy control problem asks for the input sequence $u_T \in \mathbb{R}^{mT}$ with minimum norm that steers the state of the system (1) from $x_0$ to $x_\mathrm{f}$ in $T$ steps. Mathematically, this is encoded in the solution of the following minimization problem:

$$\begin{aligned} \min_{u_T} \quad & \|u_T\|_2^2, \\ \text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t), \\ & x(0) = x_0, \; x(T) = x_\mathrm{f}. \end{aligned} \tag{11}$$

As a classic result [20], the minimization problem (11) is feasible if and only if $x_\mathrm{f}$ is reachable in $T$-steps from $x_0$, or, equivalently, if and only if $(x_\mathrm{f} - A^T x_0) \in \mathrm{Im}(C_T)$, where $C_T$ is the $T$-steps controllability matrix of the system. In this case, the solution to (11) is unique and can be computed as

$$u_T^* = C_T^\dagger (x_\mathrm{f} - A^T x_0). \tag{12}$$

In the remaining of this section, we will derive closed-form expressions of $u_T^*$ based on the dataset $\mathcal{D}$ without relying on the identification of the system matrices $A$ and $B$. To this end, we will make use of the following assumptions:

(A1) The state $x_\mathrm{f}$ is reachable in $T$-steps from the state $x_0$.

(A2) The dataset $\mathcal{D}$ contains (possibly repeated) indices $k_1, \ldots, k_\ell \in \{1, \ldots, M\}$ such that $\sum_{i=1}^{\ell} T_{k_i} = T$.

### B. Data-driven expressions of minimum energy controls

Let $k_1, \ldots, k_\ell \in \{1, \ldots, M\}$ be such that $\sum_{i=1}^{\ell} T_{k_i} = T$, and consider the following minimization problem:

$$\begin{aligned} \min_{\alpha} \quad & \|G\alpha\|_2^2 \\ \text{s.t.} \quad & \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix} = \bar{H}\alpha, \end{aligned} \tag{13}$$

where $\alpha \in \mathbb{R}^{q_{k_\ell} + \cdots + q_{k_1} + n}$ is the optimization variable, $q_{k_i} = \dim \mathrm{Ker}(X_{0,k_i})$, $G$ is as in (4), and $\bar{H}$ is the matrix comprising the first and last (row) block of $H$ in (5), namely:

$$\bar{H} = \begin{bmatrix} 0 & \cdots & 0 & 0 & I \\ \tilde{X}_\ell & \cdots & \prod_{i=0}^{\ell-3} Q_{\ell-i} \tilde{X}_2 & \prod_{i=0}^{\ell-2} Q_{\ell-i} \tilde{X}_1 & \prod_{i=0}^{\ell-1} Q_{\ell-i} \end{bmatrix}. \tag{14}$$

The next theorem shows that the solution to (13) leads to a data-driven expression of the $T$-steps minimum-energy control input from $x_0$ to $x_\mathrm{f}$ for the system (1).

*Theorem 4.1:* ***(Data-driven minimum-energy controls)*** Assume that $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1, \ldots, \ell\}$. The $T$-steps minimum-energy control input to drive the system (1) from $x_0$ to $x_\mathrm{f}$ can be expressed as

$$u_T^* = (I - GK_{\bar{H}}(GK_{\bar{H}})^\dagger) G \bar{H}^\dagger \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix}. \tag{15}$$

*Proof:* Since $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ has full row rank for all $i \in \{1, \ldots, \ell\}$ and $x_\mathrm{f}$ is reachable in $T$ steps from $x_0$ by assumption, Theorem 3.1 ensures that there exists a real vector $\alpha^*$ satisfying

$$u_T^* = G\alpha^* \quad \text{and} \quad \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix} = \bar{H}\alpha^*.$$

Because the $T$-steps minimum-energy control input $u_T^* = G\alpha^*$ is unique, $\alpha^*$ is also a solution to problem (13), and its computation is equivalent to computing $u_T^*$. By direct calculation, any solution to problem (13) has the form

$$\alpha^* = (\bar{H}^\dagger - K_{\bar{H}}(GK_{\bar{H}})^\dagger G\bar{H}^\dagger)\begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix} + g,$$

where $g$ is an arbitrary vector belonging to the kernel of $G$. Finally, by substituting the above expression of $\alpha^*$ in $u_T^* = G\alpha^*$, the data-driven expression (15) directly follows. ■

Theorem 4.1 exploits the solution to the optimization problem (13) and the data-based representation of sampled system trajectories established in Theorem 3.1 to compute a closed-form data-driven expression of the minimum-energy input $u_T^*$ based on the dataset $\mathcal{D}$. Alternatively, a data-based expression of $u_T^*$ can be derived via estimation of the $T$-steps controllability matrix $C_T$ and matrix $A^T$, as we show next.

*Theorem 4.2:* ***(Alternative expression of data-driven minimum-energy controls)*** Assume that $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1,\dots,\ell\}$. The $T$-steps minimum-energy input to drive (1) from $x_0$ to $x_\mathrm{f}$ can be expressed as

$$u_T^* = \hat{C}_T^\dagger \begin{bmatrix} -\prod_{i=0}^{\ell-1} Q_{\ell-i} & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix}, \tag{16}$$

where, for all $i \in \{1,\dots,\ell\}$,

$$\begin{aligned} \hat{C}_T &= \begin{bmatrix} L_\ell & Q_\ell L_{\ell-1} & \cdots & \prod_{i=0}^{\ell-2} Q_{\ell-i} L_1 \end{bmatrix}, \\ Q_i &= X_{k_i} K_{U_{k_i}} (X_{0,k_i} K_{U_{k_i}})^\dagger, \text{ and} \\ L_i &= X_{k_i} K_{X_{0,k_i}} (U_{k_i} K_{X_{0,k_i}})^\dagger. \end{aligned} \tag{17}$$

*Proof:* Notice that

$$\begin{aligned} X_{k_i} K_{U_{k_i}} &= (A^{T_{k_i}} X_{0,k_i} + C_{T_{k_i}} U_{k_i}) K_{U_{k_i}} \\ &= A^{T_{k_i}} X_{0,k_i} K_{U_{k_i}}. \end{aligned}$$

Because $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ has full row rank for all $i$, $X_{0,k_i} K_{U_{k_i}}$ has also full row rank for all $i$, so that it holds

$$Q_i = X_{k_i} K_{U_{k_i}} (X_{0,k_i} K_{U_{k_i}})^\dagger = A^{T_{k_i}}. \tag{18}$$

Similarly, notice that

$$\begin{aligned} X_{k_i} K_{X_{0,k_i}} &= (A^{T_{k_i}} X_{0,k_i} + C_{T_{k_i}} U_{k_i}) K_{X_{0,k_i}}, \\ &= C_{T_{k_i}} U_{k_i} K_{X_{0,k_i}}, \end{aligned}$$

and, because $U_{k_i} K_{X_{0,k_i}}$ has full row rank for all $i$, we have

$$L_i = X_{k_i} K_{X_{0,k_i}} (U_{k_i} K_{X_{0,k_i}})^\dagger = C_{T_i}. \tag{19}$$

From (18) and (19), it follows that $\hat{C}_T = C_T$ and $\prod_{i=0}^{\ell-1} Q_{\ell-i} = A^T$. Finally, since, by assumption, $x_\mathrm{f}$ is reachable in $T$ steps from $x_0$, the data-driven expression (16) directly follows from the model-based expression (12). ■

In Fig. 1 we compare the numerical performance of the model-based input (12) and our data-driven expressions (15) and (16) for a system of dimension $n = 20$, a number of inputs $m = 2$, and randomly generated data consisting of $M = 4$ datasets featuring different time horizons. Each dataset contains an identical number of data $N$. For values of $N$ in the gray region, the kernel of every data matrix $X_{0,i}$ and $U_i$ is empty and, therefore, the data-driven inputs (15) and (16) are zero. As soon as $N$ equals the number of rows of the largest matrix $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ ($N = 32$ in the figure), the norm of the data-driven inputs reaches the optimal one (Fig. 1(a)), and the corresponding error in the final state rapidly decays to zero (Fig. 1(b)), in agreement with Theorems 4.1 and 4.2.

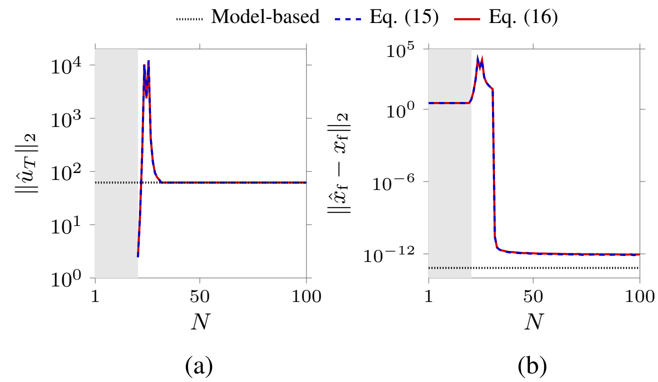



Fig. 1. In this figure we compare the numerical performance of the model-based minimum-energy input (12) and the data-driven minimum-energy inputs (15) and (16). We choose a system of dimension $n = 20$ with $m = 2$ inputs. The system matrices $A$ and $B$ have been generated randomly with i.i.d. normal entries. Data have been divided into $M = 4$ datasets with time horizons $T_i = 2 + i$, $i = 1,\dots,M$. The $i$-th dataset, $i = 1,\dots,M$, contains $N_i = N$ measurements. We choose a control horizon $T = \sum_{i=1}^M T_i = 18$. The entries of the data matrices $X_{0,i}$ and $U_i$, initial state $x_0$, and final state $x_\mathrm{f}$ have been independently drawn from of a normal distribution. The plots show the norm of the minimum-energy input (panel (a)) and the corresponding error in the final state (panel (b)) for the model-based expression (12) and the data-driven expressions (15) and (16) as a function of the size of the datasets $N$. For the data-driven input (15) we replace $(GK_{\bar{H}})^\dagger$ with $(GK_{\bar{H}})_\varepsilon^\dagger$ where $\varepsilon = 10^{-8}$ (cf. Remark 2). All curves concerning the data-driven strategies represent the average over 500 random realizations of the data matrices. In the gray regions the data-driven inputs are zero since the kernel of every matrix $X_{0,i}$ and $U_i$ is empty.

*Remark 1:* ***(Minimum number of required experiments)*** Theorems 4.1 and 4.2 provide exact data-driven expressions of the $T$-steps minimum-energy control input from $x_0$ to $x_\mathrm{f}$, under the assumption that the data matrix $[X_{0,k_i}^\mathsf{T} \; U_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1,\dots,\ell\}$. For this condition to be satisfied, at least $N_i = T_{k_i} m + n$ experiments must be collected for each control time $T_{k_i}$. If there exists $j \in \{1,\dots,M\}$ such that $T_{k_j} = 1$ (unit-length data), $m + n$ measurements suffice to reconstruct the $T$-steps minimum-energy control input, for every horizon $T$. In this case, our expressions implicitly estimate the system matrices $A$ and $B$. Specifically, in (15) and (16), $Q_j = A$, and, in (16), $L_j = B$. Hence, in this case, using our data-driven expressions or a sequential system identification and control design approach seem to be equivalent from a computational viewpoint. □

*Remark 2:* ***(Numerical properties of*** (15) ***and*** (16)***)*** While the data-driven expression (16) appears to be numerically stable (i.e., small numerical errors yield small deviations from the minimum-energy control), (15) suffers from numerical instabilities. Precisely, in the case of small numerical errors, the (row) rank of matrix $GK_{\bar{H}}$ could become full, yielding $u_T^* = 0$ in (15) regardless of the value of $x_0$ and $x_\mathrm{f}$. To remedy this situation, it is numerically convenient to replace $(GK_{\bar{H}})^\dagger$ in (15) with $(GK_{\bar{H}})_\varepsilon^\dagger$, where $(A)_\varepsilon^\dagger$ denotes the Moore–Penrose pseudoinverse of $A$ that treats as zero the singular values of $A$ that are smaller than $\varepsilon > 0$. As a rule of thumb, $\varepsilon$ should be set to a value slightly larger than the expected magnitude of the numerical errors.□

## V. Data-driven minimum-energy controls with noisy data

In this section, we assume that the dataset $\mathcal{D}$ is corrupted by additive i.i.d. noise with known second-order statistics. Specifically, for all $i \in \{1, \dots, M\}$, we consider corrupted data matrices of the form

$$U_i = \bar{U}_i + W_{U_i}, X_{0,i} = \bar{X}_{0,i} + W_{X_{0,i}}, X_i = \bar{X}_i + W_{X_i}, \tag{20}$$

where $\bar{U}_i$, $\bar{X}_{0,i}$, and $\bar{X}_i$ denote the true data matrices, and the entries of $W_{U_i}$, $W_{X_{0,i}}$, and $W_{X_i}$ are i.i.d. random variables with zero mean and variance $\sigma^2_{U_i}$, $\sigma^2_{X_{0,i}}$, and $\sigma^2_{X_i}$, respectively. In this case, the data-driven expressions (15) and (16) are typically biased (see [6, Remark 3] for an explicit example in a simplified scenario), yielding incorrect control inputs even when the number of data grows unbounded. In this section, we will show that the effect of noise can be cancelled, in the limit of infinite data, by suitably "correcting" these expressions. Specifically, inspired by [21], we will introduce correction terms that compensate for the variance-dependent terms generated by the pseudoinverse and kernel operations in (15) and (16), leading to asymptotically correct (or, equivalently, *consistent*) data-driven expressions.[4]

We consider first the data-driven expression (16), and rewrite the terms $Q_i$, $L_i$ in (17), respectively, as

$$\begin{aligned} Q_i &= X_{k_i}\Pi_{U_{k_i}}X_{0,k_i}^\top(X_{0,k_i}\Pi_{U_{k_i}}X_{0,k_i}^\top)^\dagger, \\ L_i &= X_{k_i}\Pi_{X_0}U_{k_i}^\top(U_{k_i}\Pi_{X_0}U_{k_i}^\top)^\dagger, \end{aligned}$$

where we used the identity $A^\dagger = A^\top(AA^\top)^\dagger$, and we replaced, without loss of generality, every term $K_AK_A^\top$ with the orthogonal projections onto $\mathrm{Ker}(A)$, $\Pi_A = I - A^\dagger A$. Next, we define the "corrected" versions of $Q_i$ and $L_i$ as

$$\begin{aligned} Q_{i,c} &= X_{k_i}\Pi_{U_{k_i},c}X_{0,k_i}^\top(X_{0,k_i}\Pi_{U_{k_i},c}X_{0,k_i}^\top - N\sigma^2_{X_0}I)^\dagger, \\ L_{i,c} &= X_{k_i}\Pi_{X_0,c}U_{k_i}^\top(U_{k_i}\Pi_{X_0,c}U_{k_i}^\top - N\sigma^2_U I)^\dagger, \end{aligned}$$

where $\Pi_{X_{0,k_i},c} = I - X_{0,k_i}^\top(X_{0,k_i}X_{0,k_i}^\top - N\sigma^2_{X_0}I)^\dagger X_{0,k_i}$ and $\Pi_{U_{k_i},c} = I - U_{k_i}^\top(U_{k_i}U_{k_i}^\top - N\sigma^2_U I)^\dagger U_{k_i}$. With these definitions in place, we introduce the following "corrected" expression of the data-driven control input (16):

$$u''_{T,c} = \hat{C}_{T,c}^\dagger \begin{bmatrix} -\prod_{i=0}^{\ell-1} Q_{\ell-i,c} & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix}, \tag{21}$$

where $\hat{C}_{T,c}$ is defined as in (17), after replacing all instances of $Q_i$ and $L_i$ with $Q_{i,c}$ and $L_{i,c}$, respectively. It is worth noting that, if only the matrices $X_i$ are affected by noise, then (21) coincides with (16), and no correction is needed.

[4]To simplify the treatment without compromising the generality of the approach, in what follows we will assume $N_i = N$, $\sigma^2_U = \sigma^2_{U_i}$, $\sigma^2_{X_0} = \sigma^2_{X_{0,i}}$, and $\sigma^2_X = \sigma^2_{X_i}$ for all $i \in \{1, \dots, M\}$.

*Theorem 5.1:* ***(Consistency of*** $u''_{T,c}$***)*** Assume that the dataset $\mathcal{D}$ is corrupted by noise as in (20), and that $[\bar{X}_{0,k_i}^\top \ \bar{U}_{k_i}^\top]^\top$ is full row rank for all $i \in \{1, \dots, \ell\}$. The data-driven control $u''_{T,c}$ in (21) converges almost surely to the minimum-energy control input $u_T^*$ as $N \to \infty$.

*Proof:* By the Strong Law of Large Numbers [22, p. 6] and the assumption on the noise, as $N \to \infty$, we have

$$\begin{aligned} \Delta_{i,1} &= \frac{1}{N}X_{0,k_i}X_{0,k_i}^\top \xrightarrow{\text{a.s.}} \frac{1}{N}\bar{X}_{0,k_i}\bar{X}_{0,k_i}^\top + \sigma^2_{X_0}I = \bar{\Delta}_{i,1}, \\ \Delta_{i,2} &= \frac{1}{N}U_{k_i}U_{k_i}^\top \xrightarrow{\text{a.s.}} \frac{1}{N}\bar{U}_{k_i}\bar{U}_{k_i}^\top + \sigma^2_U I = \bar{\Delta}_{i,2}, \\ \Delta_{i,3} &= \frac{1}{N}X_{k_i}X_{0,k_i}^\top \xrightarrow{\text{a.s.}} \frac{1}{N}\bar{X}_{k_i}\bar{X}_{0,k_i}^\top = \bar{\Delta}_{i,3}, \\ \Delta_{i,4} &= \frac{1}{N}X_{k_i}U_{k_i}^\top \xrightarrow{\text{a.s.}} \frac{1}{N}\bar{X}_{k_i}\bar{U}_{k_i}^\top = \bar{\Delta}_{i,4}, \end{aligned} \tag{22}$$

where $\xrightarrow{\text{a.s.}}$ denotes almost sure convergence. Each matrix $Q_{i,c}$ can be written as a function of $\Delta_{i,j}$, $j = 1, 2, 3$, namely,

$$\begin{aligned} Q_{i,c} = {} & (\Delta_{i,3} - \Delta_{i,3}(\Delta_{i,2} - \sigma^2_{U_{k_i}}I)^\dagger\Delta_{i,3}) \cdot \\ & \cdot (\Delta_{i,1} - \sigma^2_{X_{0,k_i}}I + \Delta_{i,1}(\Delta_{i,2} - \sigma^2_{U_{k_i}}I)^\dagger\Delta_{i,1})^\dagger. \end{aligned}$$

Further, notice that $Q_{i,c}$ is continuous at $\Delta_{i,j} = \bar{\Delta}_{i,j}$, $j = 1, 2, 3$, since $[\bar{X}_{0,k_i}^\top \ \bar{U}_{k_i}^\top]^\top$ is full row rank by assumption. Thus, by (22) and the Continuous Mapping Theorem [22, Theorem 2.3], as $N \to \infty$,

$$Q_{i,c} \xrightarrow{\text{a.s.}} \bar{Q}_i, \tag{23}$$

where $\bar{Q}_i = \bar{X}_{k_i}K_{\bar{U}_{k_i}}(\bar{X}_{0,k_i}K_{\bar{U}_{k_i}})^\dagger$. Analogously, each $L_{i,c}$ can be written as

$$\begin{aligned} L_{i,c} = {} & (\Delta_{i,4} - \Delta_{i,4}(\Delta_{i,1} - \sigma^2_{X_{0,k_i}}I)^\dagger\Delta_{i,4}) \cdot \\ & \cdot (\Delta_{i,2} - \sigma^2_{U_{k_i}}I + \Delta_{i,2}(\Delta_{i,1} - \sigma^2_{X_{0,k_i}}I)^\dagger\Delta_{i,2})^\dagger, \end{aligned}$$

and the same argument as before shows that, as $N \to \infty$,

$$L_{i,c} \xrightarrow{\text{a.s.}} \bar{L}_i, \tag{24}$$

where $\bar{L}_i = \bar{X}_{k_i}K_{\bar{X}_{0,k_i}}(\bar{U}_{k_i}K_{\bar{X}_{0,k_i}})^\dagger$. Finally, by applying (23), (24), and, once again, the Continuous Mapping Theorem, we conclude that $u''_{T,c} \xrightarrow{\text{a.s.}} u_T^*$ as $N \to \infty$. ■

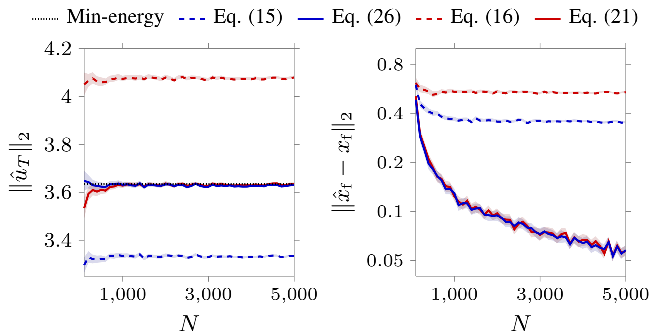
Consider now the data-driven control (15). After some algebraic manipulations, it can be rewritten as

$$u_T^* = (I - G\Pi_{\bar{H}}G^\top(G\Pi_{\bar{H}}G^\top)^\dagger)G\bar{H}^\top(\bar{H}\bar{H}^\top)^\dagger \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix}. \tag{25}$$

We introduce the following "corrected" version of (25):

$$\begin{aligned} u'_{T,c} = {} & (I - (G_c\Pi_{\bar{H},c}G_c^\top - N\sigma^2_U I)(G_c\Pi_{\bar{H},c}G_c^\top - N\sigma^2_U I)_\varepsilon^\dagger) \cdot \\ & \cdot G_c\bar{H}_c^\top(\bar{H}_c\bar{H}_c^\top - \Delta_{\bar{H}})^\dagger \begin{bmatrix} x_0 \\ x_\mathrm{f} \end{bmatrix}, \end{aligned} \tag{26}$$

where $G_c$ and $\bar{H}_c$ are defined as $G$ and $\bar{H}$, after replacing all instances of $Q_i$ and $K_{X_{0,k_i}}$ with $Q_{i,c}$ and $\Pi_{X_{0,k_i},c}$,




(a) (b)

Fig. 2. In this figure we compare the behavior of the data-driven minimum-energy inputs (15) and (16) and their corrected versions (26) and (21), respectively. We choose a system of dimension $n = 4$ with $m = 2$ inputs. The system matrices $A$ and $B$ have been generated randomly with i.i.d. normal entries. Data have been divided into $M = 2$ datasets with time horizons $T_1 = 3$ and $T_2 = 4$ and $N_1 = N_2 = N$ measurements. We choose a control horizon $T = T_1 + T_2 = 7$. The entries of every $X_{0,i}$ and $U_i$ are independently and uniformly distributed in $[0, 1]$. The entries of the initial state $x_0$ and final state $x_f$ have been independently drawn from of a normal distribution. The plots show the norm of the minimum-energy input (panel (a)) and the corresponding error in the final state (panel (b)) for all the data-driven expressions as a function of the number of data $N$. For the data-driven inputs (15) (cf. Remark 2) and (26) we choose a tolerance $\varepsilon = 10^{-8}$. The entries of all data matrices are corrupted by i.i.d. Gaussian noise as in (20) with variance $\sigma_X^2 = \sigma_{X_0}^2 = \sigma_U^2 = 0.1$. The solid and dashed curves represent the average over 100 realizations of the noise, whereas the light-colored regions denote the 95% confidence intervals around the mean.

respectively, the operation $(\cdot)^\dagger_\varepsilon$ is defined in Remark 2, and

$$\Pi_{\bar{H},c} = I - \bar{H}_c^\mathsf{T}(\bar{H}_c\bar{H}_c^\mathsf{T} - \Delta_{\bar{H}})^\dagger \bar{H}_c, \ \Delta_{\bar{H}} = \begin{bmatrix} 0_n & 0_n \\ 0_n & \Delta_{\bar{H},2} \end{bmatrix},$$

$$\Delta_{\bar{H},2} = N\sigma_X^2 \sum_{j=0}^{\ell} \prod_{i=0}^{j-1} Q_{\ell-i,c} \left( \prod_{i=0}^{j-1} Q_{\ell-i,c} \right)^\mathsf{T}, \ Q_{0,c} = I.$$

*Theorem 5.2:* ***(Consistency of*** $u'_{T,c}$***)*** Assume that $\mathcal{D}$ is corrupted by noise as in (20), and that $[\bar{X}_{0,k_i}^\mathsf{T} \ \bar{U}_{k_i}^\mathsf{T}]^\mathsf{T}$ is full row rank for all $i \in \{1, \dots, \ell\}$. For $\varepsilon > 0$ sufficiently small, the data-driven control $u'_{T,c}$ in (26) converges almost surely to the minimum-energy control input $u_T^*$ as $N \to \infty$.

The proof of Theorem 5.2 follows closely the one of Theorem 5.1 and is therefore omitted. In Fig. 2, we illustrate the behavior of the data-driven expressions (15) and (16), and their corrected versions (26) and (21), respectively, as a function of the data size $N$. Each dataset is corrupted by i.i.d. Gaussian noise as in (20) with $\sigma_X^2 = \sigma_{X_0}^2 = \sigma_U^2 = 0.01$. As the number of data $N$ increases, the corrected data-driven expressions (26) and (21) approach the minimum-energy cost (Fig. 2(a)) and the corresponding errors in the final state decrease (Fig. 2(b)), as predicted by Theorems 5.1 and 5.2.

## VI. Conclusion

In this paper we address the problem of computing minimum-energy controls for linear systems using heterogeneous data. Specifically, we consider data consisting of input-state trajectories featuring different time horizons and initial conditions. We derive two different data-driven expressions of minimum-energy controls for a wide range of control horizons, possibly different from those in the experiments. When data are affected by i.i.d. noise with zero mean and known variance, we modify our expressions so to ensure convergence to the correct controls in the limit of infinite data.

Directions for future work include the application of our approach and data collection setting to other control problems, such as LQR and MPC, the sensitivity analysis of the corrected data-driven control inputs to uncertainty in the noise variances, and the derivation of non-asymptotic bounds on the reconstruction error in the case of finite noisy data.

## References


[1] C. De Persis and P. Tesi. Formulas for data-driven control: Stabilization, optimality and robustness. *IEEE Transactions on Automatic Control*, 65(3):909–924, 2020.

[2] J. Coulson, J. Lygeros, and F. Dörfler. Data-enabled predictive control: In the shallows of the DeePC. In *2019 18th European Control Conference (ECC)*, pages 307–312, 2019.

[3] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel. Data informativity: a new perspective on data-driven analysis and control. *arXiv preprint arXiv:1908.00468*, 2019.

[4] J. Berberich and F. Allgöwer. A trajectory-based framework for data-driven system analysis and control. *arXiv preprint arXiv:1903.10723*, 2019.

[5] G. R. Gonçalves da Silva, A. S. Bazanella, C. Lorenzini, and L. Campestrini. Data-driven LQR control design. *IEEE Control Systems Letters*, 3(1):180–185, 2019.

[6] G. Baggio, V. Katewa, and F. Pasqualetti. Data-driven minimum-energy controls for linear systems. *IEEE Control Systems Letters*, 3(3):589–594, 2019.

[7] Z.-S. Hou and Z. Wang. From model-based control to data-driven control: Survey, classification and perspective. *Information Sciences*, 235:3–35, 2013.

[8] B. Recht. A tour of reinforcement learning: The view from continuous control. *Annual Review of Control, Robotics, and Autonomous Systems*, 2018.

[9] D. A. Bristow, M. Tharayil, and A. G. Alleyne. A survey of iterative learning control. *IEEE control systems magazine*, 26(3):96–114, 2006.

[10] K. J. Åström and B. Wittenmark. On self tuning regulators. *Automatica*, 9(2):185–199, 1973.

[11] I. Markovsky and P. Rapisarda. Data-driven simulation and control. *International Journal of Control*, 81(12):1946–1959, 2008.

[12] F. Pasqualetti, S. Zampieri, and F. Bullo. Controllability metrics, limitations and algorithms for complex networks. *IEEE Transactions on Control of Network Systems*, 1(1):40–52, 2014.

[13] T. H. Summers, F. L. Cortesi, and J. Lygeros. On submodularity and controllability in complex dynamical networks. *IEEE Transactions on Control of Network Systems*, 3(1):91–101, 2016.

[14] N. Bof, G. Baggio, and S. Zampieri. On the role of network centrality in the controllability of complex networks. *IEEE Transactions on Control of Network Systems*, 4(3):643–653, 2017.

[15] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.

[16] P. Van Overschee and B. L. De Moor. *Subspace identification for linear systems: Theory-Implementation-Applications*. Springer US, 1996.

[17] I. Markovsky. A missing data approach to data-driven filtering and control. *IEEE Transactions on Automatic Control*, 62(4):1972–1978, 2017.

[18] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu. On the sample complexity of the linear quadratic regulator. *arXiv preprint arXiv:1710.01688*, 2018.

[19] I. Markovsky, J. C. Willems, P. Rapisarda, and B. L. M. De Moor. Algorithms for deterministic balanced subspace identification. *Automatica*, 41(5):755–766, 2005.

[20] T. Kailath. *Linear Systems*. Prentice-Hall, 1980.

[21] I. Markovsky, R. J. Vaccaro, and D. Van Huffel. System identification by optimal subspace estimation. Technical Report 06–162, Dept. EE, KU Leuven, 2006.

[22] A. W. Van der Vaart. *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, 2000.