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Network Composition for Optimal Disturbance Rejection

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***Abstract*— This paper investigates how the topology of a dynamical network affects its robustness against exogenous disturbances. We consider Laplacian-based network dynamics, and we adopt the $\mathcal{H}_2$ system norm to measure the robustness of the network against disturbances. For networks arising from the composition of atomic structures, we provide a closed-form expression of the robustness against disturbances, and we identify optimal composition rules. Specifically, we show that networks consisting of multiple atomic structures are less robust than each isolated part, and that robust structures arise by interconnecting nodes of the atomic components with highest degree. Finally, we describe an algorithm for the design of robust composite networks.**

I. INTRODUCTION

Network systems are ubiquitous in engineering, social, and natural domains, where they enable complex functionalities by interconnecting diverse components. An important property of such systems is their robustness to external disturbances altering individual nodes or interconnection dynamics: the failure of a single network component may cascade into the failure of all interconnected parts [1].

Robustness of an interconnected network depends on both the robustness of the isolated subnetworks, as well as on the topological properties of the interconnection structure. In this paper we propose a mathematical framework to characterize the robustness of interconnected network systems with respect to the interconnection structure. We measure robustness of a network based on its $\mathcal{H}_2$ norm, that is, based on the effect on the network nodes of a white noise disturbance. For our metric, we show that interconnected networks are less robust than the isolated components, and that certain nodes, the nodes of the atomic components with highest degree, enable more robust interconnection of networks. Our results are in accordance and provide a quantitative study of recent findings; e.g., see [1].

Related work The majority of the existing research on the robustness of dynamical systems and networks focuses on single or isolated components. Classic work in the controls literature defines different measures for the robustness of a dynamical system to disturbances; e.g., see [2]. In the context of network systems, network re-wiring and re-weighting schemes are proposed in [3], [4] to improve the robustness of a single network to environmental disturbances. In this paper we improve the results upon these directions by considering the robustness of interconnected networks with respect to the interconnection topology.

In the more recent literature on network of networks, different metrics have been used to analyze the robustness of interconnected systems. In [5], cascading failures through interconnected networks are studied via percolation theory. In [6], robustness against random failures or intentional attacks is considered, and a block-based model is proposed to incorporate information of both connectivity and correlations among blocks and links, and infer upon the structure of robust networks. Multi-layer networks, their dynamical properties, and their robustness to random failures are studied, for instance, in [7], [8]. Finally, the importance of the interconnection topology and its structural properties to mitigate failures across networks is highlighted in [9]. We depart from these works by considering a different measure of network robustness and network dynamics, by providing a control-theoretic characterization of the robustness of interconnected networks, and by providing an algorithm for the design of optimally robust networks of networks.

Paper contributions The contributions of this paper are threefold. First, we construct a mathematical framework to analyze the robustness of interconnected network systems, where network systems evolve according to modified Laplacian dynamics. We adopt the $\mathcal{H}_2$ system norm to quantify the robustness of a network system to external disturbances. For the case of two interconnected networks, we provide a closed-form expression of the $\mathcal{H}_2$ system norm with respect to the individual components. We show, and quantify, that the $\mathcal{H}_2$ system norm always increases upon interconnection of multiple blocks, so that interconnected networks are less robust than the isolated parts. Second, we prove that interconnections among nodes of the atomic networks with highest degree yield maximum robustness of the interconnected system. In other words, we provide a network interconnection rule that maximizes robustness to disturbances. Third and finally, we describe an interconnection algorithm for the case of multiple subnetworks, and we provide bounds on the robustness of the composite network.

II. PROBLEM SETUP AND PRELIMINARY NOTIONS

Let $\mathcal{S} = \{s_1, \dots, s_n\}$ be a set of n *atomic* dynamical networks. Every network is described by the connected and undirected graph $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, with $|\mathcal{V}_i| = n_i$. Let the dynamics of the network s_i be described by

$$\dot{x}_i = -\mathcal{Q}_i\, x_i, \quad \mathcal{Q}_i = \alpha\, \mathcal{I}_i + \mathcal{L}_i. \tag{1}$$

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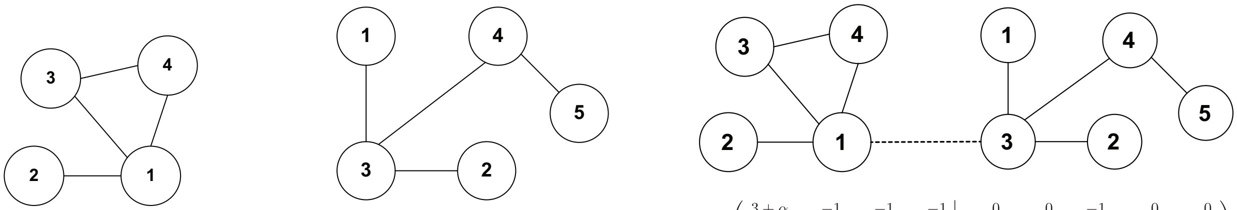

$$Q_1 = \begin{pmatrix} 3+\alpha & -1 & -1 & -1 \\ -1 & 1+\alpha & 0 & 0 \\ -1 & 0 & 2+\alpha & -1 \\ -1 & 0 & -1 & 2+\alpha \end{pmatrix}$$

(a)

$$Q_2 = \begin{pmatrix} 1+\alpha & 0 & -1 & 0 & 0 \\ 0 & 1+\alpha & -1 & 0 & 0 \\ -1 & -1 & 3+\alpha & -1 & 0 \\ 0 & 0 & -1 & 2+\alpha & -1 \\ 0 & 0 & 0 & -1 & 1+\alpha \end{pmatrix}$$

(b)

$$Q_{12} = \left(\begin{array}{cccc|ccccc} 3+\alpha & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1+\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2+\alpha & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2+\alpha & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1+\alpha & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\alpha & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 3+\alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\alpha & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1+\alpha \end{array}\right)$$

(c)

Fig. 1. Example of network composition. In particular, Figure 1-a) and Figure 1-b) show two dynamical networks $\mathcal{G}_1$ and $\mathcal{G}_2$ along with their network matrices $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively. Figure 1-c) shows the graph $\mathcal{G}_{12}$ resulting from the interconnection (dashed edge) of two nodes (hubs), and the related network matrix $\mathcal{Q}_{12}$.

where $x_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_i}$ is the map containing the state of the i-th network, $\mathcal{L}_i$ is the Laplacian matrix associated with $\mathcal{G}_i$ [10], and $\alpha \in \mathbb{N}_{>1}$. The network dynamics (1) can be thought as the composition of two parts: the *nominal* dynamics, i.e., $\mathcal{I} + \mathcal{L}$, and the network *interconnection* dynamics, i.e., $(\alpha - 1)\mathcal{I}$. The parameter $\alpha > 1$ represents an upper bound on the number of interconnections that can be performed through each node of the network. Notice that, by construction, $\mathcal{Q}_i$ is positive definite and strictly diagonally dominant, hence invertible [11].

We adopt the open loop $\mathcal{H}_2$ system norm to measure the ability of a network to reject disturbances [3]. As we are interested in quantifying the effect on the whole state of a disturbance affecting all network nodes, the $\mathcal{H}_2$ of the network s_i is defined as

$$\mathcal{H}_2(s_i) = \text{Trace}\left(\int_0^\infty e^{-2\mathcal{Q}_i t}\, dt\right) = \frac{1}{2}\text{Trace}(\mathcal{Q}_i^{-1}).$$

In order to interconnect the networks s_i and s_j, we select two nodes $h \in \mathcal{V}_i$ and $k \in \mathcal{V}_j$ for which the constraint α on the maximum number of interconnection is satisfied,[1] and define the composite network s_{ij} as $\mathcal{G}_{ij} = (\mathcal{V}_{ij}, \mathcal{E}_{ij})$, where $\mathcal{V}_{ij} = \mathcal{V}_i \cup \mathcal{V}_j$ and $\mathcal{E}_{ij} = \mathcal{E}_i \cup \mathcal{E}_j \cup (h, k)$. The dynamics and the matrix of the composite network are defined as

$$\dot{x}_{ij} = -\mathcal{Q}_{ij}\, x_{ij}, \quad \mathcal{Q}_{ij} = \begin{bmatrix} \mathcal{Q}_i & -e_h e_k^\mathsf{T} \\ -e_k e_h^\mathsf{T} & \mathcal{Q}_j \end{bmatrix}. \qquad (2)$$

where $x_{ij} = [x_i^\mathsf{T}\ x_j^\mathsf{T}]^\mathsf{T}$, and e_i is the i-th canonical vector of appropriate dimension. Clearly, $\mathcal{H}_2(s_{ij}) = \text{Trace}(\mathcal{Q}_{ij}^{-1})/2$. By constraining each network to perform at most $\alpha - 1$ interconnections through each of its nodes, we ensure that the $\mathcal{Q}_{ij}$ matrix remains positive definite and strictly diagonally dominant, and hence invertible.

III. Optimal Interconnection of Networks

In this section we characterize how the $\mathcal{H}_2$ norm changes when multiple networks are interconnected. We start with two networks, and then generalize our results to the case of multiple networks

A. Interconnection of Two Networks

We now consider two atomic networks $\mathcal{Q}_i$ and Q_j, and provide a closed-form expression for the $\mathcal{H}_2$ norm of the interconnected network $\mathcal{Q}_{ij}$. To this aim, let $\mathcal{Q}(h, k)$ and $\mathcal{Q}^{-1}(h, k)$ denote the entry in the h-th row and k-th column for the matrix $\mathcal{Q}$ and the inverse matrix $\mathcal{Q}^{-1}$, respectively. In addition, let $\mathcal{Q}^{-1}(h, :)$ and $\mathcal{Q}^{-1}(:, k)$ denote the h-th row and the k-th column of the matrix $\mathcal{Q}^{-1}$, respectively.

Theorem 1: ***($\mathcal{H}_2$ norm of two interconnected networks)*** Let $\mathcal{Q}_{ij}$ be as in (2). Then,

$$\text{Trace}(\mathcal{Q}_{ij}^{-1}) = \text{Trace}(\mathcal{Q}_i^{-1}) + \text{Trace}(\mathcal{Q}_j^{-1}) + \lambda_{ij}^{hk} + \lambda_{ij}^{kh},$$

where

$$\begin{aligned} \lambda_{ij}^{hk} &= \frac{\|\mathcal{Q}_i^{-1}(:, h)\|^2}{1/\mathcal{Q}_j^{-1}(k, k) - \mathcal{Q}_i^{-1}(h, h)} > 0, \text{ and} \\ \lambda_{ij}^{kh} &= \frac{\|\mathcal{Q}_j^{-1}(:, k)\|^2}{1/\mathcal{Q}_i^{-1}(h, h) - \mathcal{Q}_j^{-1}(k, k)} > 0. \end{aligned} \qquad (3)$$

Proof: See Appendix B. ■

Theorem 1 provides a general closed-form expression for the $\mathcal{H}_2$ norm of a composite network. It should be noticed that the provided relation depends on the interconnection parameter α, implicitly considered in $\mathcal{Q}_i$ and $\mathcal{Q}_j$. Theorem 1 also implies that networks arising from the interconnection of two isolated atomic networks are less robust than the isolated components. In fact, $\text{Trace}(\mathcal{Q}_{ij}^{-1}) > \text{Trace}(\mathcal{Q}_i^{-1}) + \text{Trace}(\mathcal{Q}_j^{-1})$. Moreover, it follows that the minimum $\mathcal{H}_2$ performance of the composite network is achieved when the interconnections nodes h and k are selected to minimize the perturbation $\lambda_{ij}^{kh} + \lambda_{ij}^{hk}$. Let $\deg(i)$ denote the degree of node i, and define *hub* a node with highest degree [10].We next show that the $\mathcal{H}_2$ norm of a composite network is minimized when h and k are hubs of the atomic networks.

[1] Given a matrix $\mathcal{Q}_i$ and a node h, to check whether an interconnection can be established through node h we simply check that the $h-th$ row-sum is greater than one, that is $\sum_j^{n_i} \mathcal{Q}_i(h, j) > 1$.

Theorem 2: ***(Connections via hubs)*** Let $\mathcal{Q}_{ij}$ be as in (2), and let h^* and k^* satisfy

$$\lambda_{ij}^{h^*k^*} + \lambda_{ij}^{k^*h^*} = \min_{h\in\mathcal{V}_i, k\in\mathcal{V}_j} \lambda_{ij}^{kh} + \lambda_{ij}^{hk},$$

where λ_{ij}^{kh} and λ_{ij}^{hk} are defined as in (3). Then,

$$\deg(h^*) = \max_{h\in\mathcal{V}_i} \deg(h), \text{ and } \deg(k^*) = \max_{k\in\mathcal{V}_j} \deg(k).$$

Proof: See Appendix C. ■

Theorem 2 implies that, to minimize the $\mathcal{H}_2$ norm, two atomic networks should be connected by creating links between nodes with highest degree. Notice that the isolated atomic networks may have multiple hubs, and the choice of an hub remains, at this stage, a combinatorial problem.

B. Interconnection of Multiple Networks

We now study the robustness of networks arising from the composition of multiple components, which themselves may already represent composite networks. We assume that at each iteration only a pairwise interconnection between two (composite) dynamical networks may be carried out.

We now introduce the following preliminary result.

Lemma 1: ***(Trace decomposition)*** Let $\mathcal{D}_i$ and $\mathcal{D}_j$ diagonal entrywise positive (integer) matrices. Let $\mathcal{A}_i = \mathcal{D}_i + \mathcal{L}_i$ and $\mathcal{A}_j = \mathcal{D}_j + \mathcal{L}_j$ be symmetric positive definite and diagonally dominant, and define be

$$\mathcal{A}_{ij} = \begin{bmatrix} \mathcal{A}_i & -e_h e_k^\mathsf{T} \\ -e_k e_h^\mathsf{T} & \mathcal{A}_j \end{bmatrix}, \tag{4}$$

for some canonical vectors e_h and e_k such that

$$h = \text{argmax}_r\, \mathcal{A}_i(r,r), \text{ and } k = \text{argmax}_r\, \mathcal{A}_j(r,r). \tag{5}$$

Then, $\mathcal{A}_{ij} = \mathcal{D}_{ij} + \mathcal{L}_{ij}$ is symmetric positive definite and diagonally dominant, and

$$\text{Trace}(\mathcal{A}_{ij}^{-1}) \le \text{Trace}(\mathcal{A}_i^{-1}) + \text{Trace}(\mathcal{A}_j^{-1}) + \Delta_{ij} + \Delta_{ji},$$

where

$$\begin{aligned} \Delta_{ij} &= \frac{(1+\gamma_i^2)^2(1+\gamma_j)^2\mathcal{A}_i^{\max}}{\gamma_i \mathcal{A}_i^{2\max}\left(16\gamma_i\gamma_j\mathcal{A}_i^{\max}A_j^{\max} - (1+\gamma_i)^2(1+\gamma_j)^2\right)}, \\ \Delta_{ji} &= \frac{(1+\gamma_i)^2(1+\gamma_j^2)^2\mathcal{A}_j^{\max}}{\gamma_j \mathcal{A}_j^{2\max}\left(16\gamma_i\gamma_j\mathcal{A}_i^{\max}A_j^{\max} - (1+\gamma_i)^2(1+\gamma_j)^2\right)}, \end{aligned} \tag{6}$$

with $\gamma_k = \lambda_{\max}(\mathcal{A}_k)/\lambda_{\min}(\mathcal{A}_k)$,

$$\mathcal{A}_k^{\max} = \max_r \mathcal{A}_k(r,r), \text{ and } \mathcal{A}_k^{2\max} = \max_r \mathcal{A}_k^2(r,r).$$

Proof: See Appendix D ■

Note that, both the the dynamical matrix of the isolated components $\mathcal{Q}_i$ in (1) and of the composite network $\mathcal{Q}_{ij}$ in (2) have the structure of the matrix $\mathcal{A}_i$ in Lemma 1. We now provide a useful result for a composite network, which relates the computation of the parameters in (6) to the isolated components. Intuitively, this will enable the (recursive) application of Lemma 1 for the derivation of robustness bounds in che case of composite networks arising from the interconnection of multiple components.

Lemma 2: ***(Bounds on composite networks)*** Let $\mathcal{Q}_i$ and $\mathcal{Q}_j$ be as in (1) or (2) and let their interconnection $\mathcal{Q}_{ij}$ be as in (4) with h and k defined in (5). Then,

$$\begin{aligned} \gamma_{ij} &< 2\max\{\mathcal{Q}_i^{\max}, \mathcal{Q}_j^{\max}\}, \\ \mathcal{Q}_{ij}^{\max} &= \max\left\{\mathcal{Q}_i^{\max}, \mathcal{Q}_j^{\max}\right\}, \\ \mathcal{Q}_{ij}^{2\max} &> \max\left\{\mathcal{Q}_i^{2\max}, \mathcal{Q}_j^{2\max}\right\}. \end{aligned}$$

Proof: See Appendix E ■

Notice that, after pairwise interconnection, the network matrix reads as in (2). Thus, with respect to the case of two interconnected networks, the isolated components on the (block) diagonal are in fact composite networks. We are now ready to state upper and lower bounds for the $\mathcal{H}_2$ norm of a composite dynamical network.

Theorem 3: ***($\mathcal{H}_2$ norm of composite networks)*** Let $\mathcal{Q}$ be a matrix resulting from the pairwise interconnection of $\mathcal{Q}_1, \dots, \mathcal{Q}_n$. Then, $\mathcal{Q}$ is positive definite and diagonally dominant, and

$$\begin{aligned} \text{Trace}(\mathcal{Q}^{-1}) &\le \sum_{i=1}^{n} \text{Trace}(\mathcal{Q}_i^{-1}) + (n-1)\bar{\Delta}^{\max}, \\ \text{Trace}(\mathcal{Q}^{-1}) &\ge \sum_{i=1}^{n} \text{Trace}(\mathcal{Q}_i^{-1}), \end{aligned}$$

where $\bar{\Delta}^{\max} = \max_{i,j=\{1,\dots,n\}}\{\bar{\Delta}_{ij} + \bar{\Delta}_{ji}\}$, with $\bar{\Delta}_{ij}$ defined as

$$\bar{\Delta}_{ij} = \frac{\left(1+(\mathcal{Q}_i^{\max})^2\right)^2(1+\mathcal{Q}_j^{\max})^2\mathcal{Q}_i^{\max}}{\gamma_i\mathcal{Q}_i^{2\max}\left(16\left(\mathcal{Q}_i^{\max}\right)^2\left(\mathcal{Q}_j^{\max}\right)^2 - \left(1+\mathcal{Q}_i^{\max}\right)^2(1+\mathcal{Q}_j^{\max})^2\right)} \tag{7}$$

Proof: See Appendix F ■

C. Numerical Results

In this section, we provide numerical results to validate our theoretical findings. In particular, motivated by the fact that the optimal interconnection between two dynamical networks is achieved via nodes with the highest degree, we propose an algorithm that, at each iteration, minimizes the perturbation due to the interconnection of hubs.[1] To evaluate the effectiveness of our algorithm, we provide a comparison against a randomized algorithm that, at each iteration, interconnects two randomly selected networks through a pair of randomly selected nodes.

We first consider a set $\mathcal{S} = \{s_1, \dots, s_n\}$ of n atomic dynamical networks, n ranging from 2 to 7 with step 1, with $n_i \in [10, 20]$ and $\alpha = 3$. For this case, we compare our algorithm against the randomized one, and against the optimal solution computed through a brute force approach. Then, we consider a set $\mathcal{S} = \{s_1, \dots, s_n\}$ of n atomic dynamical networks, n ranging from 10 to 50 with step 10. Due to the dimension of the problem, in this second case we compare our algorithm against the randomized one only. In both cases, for each n, we generate 100 set of networks.

Figure 2 shows the outcome for the first set of simulations, where the x-axis represents the number of networks

[1] For a composite network nodes are labeled as hubs according to their role in the atomic network they originally belong to.

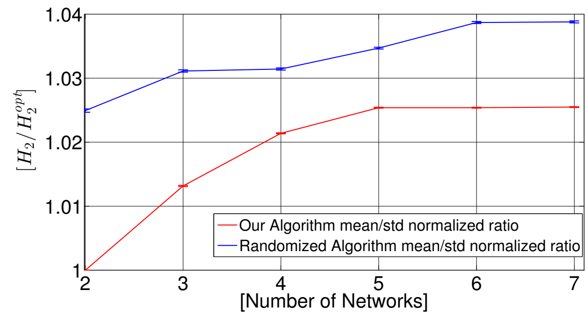

Fig. 2. Simulation results for the first set (mean and standard deviation) over 100 run. The red lines represent the proposed algorithm, the randomized one is depicted in blue.

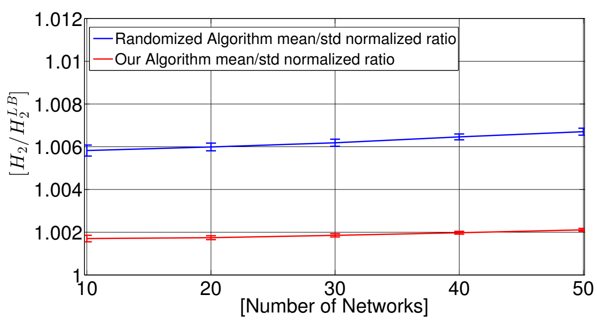

Fig. 3. Simulation results for the second set (mean and standard deviation) over 100 set of $\mathcal{S}$ networks. The red lines represent the proposed algorithm, the randomized one is depicted in blue.

involved and the y-axis represents the normalized ratio between the $\mathcal{H}_2$ norm of the composite network and the $\mathcal{H}_2$ norm of the optimal solution, i.e., $\mathcal{H}_2^{\text{opt}}$. In particular, both the mean value and the standard deviation over the 100 run were computed for both our algorithm and the randomized one. According to the numerical results, our algorithm always provides a smaller gap with respected to the the optimal solution compared to the randomized one.

Figure 3 shows the outcome for the second set of simulations, where the x-axis represents the number of networks involved and the y-axis represents the normalized ratio between the $\mathcal{H}_2$ norm of the composite network and the $\mathcal{H}_2$ norm of the lower bound computed according to Theorem 3, i.e., $\mathcal{H}_2^{\mathcal{LB}}$. Also for this numerical evaluation, both the mean value and the standard deviation over the 100 run were computed for both our algorithm and the randomized one. According to the numerical results, also in this case our algorithm always provides a smaller gap with respected to the lower bound compared to the randomized one.

For the second set of simulations, Table I also provides the value of the upper bound computed according to Theorem 3. It can be noticed that the upper bound $\mathcal{UB}$ is not tight. This can be explained by the looseness of the bound given in Lemma 2 for the terms γ_{ij}.

IV. CONCLUSION

In this paper we characterize the robustness of interconnected networks as a function of the interconnection topology. We consider networks with Laplacian-based dynamics

TABLE I

UPPER BOUND FOR THE SECOND SET OF SIMULATIONS

Networks	10	20	30	40	50
Upp. Bound	1.464	1.492	1.498	1.506	1.509
Random	1.0058	1.0060	1.0062	1.0065	1.0067
Our	1.0017	1.0017	1.0019	1.0020	1.0021

and quantify that interconnected networks are always less robust than the isolated components. Further, for the interconnection of two networks and for the multiple networks problem under pairwise connection scheme, we show that interconnections among nodes of the atomic components with highest degree yield maximum robustness. Finally, we propose an interconnection rule for the design of robust composite networks, and validate its effectiveness through simulations. Several directions are left for future work, including the extension to general network dynamics where different nodes feature different interconnection capabilities.

APPENDIX

A. Proof Preliminaries

In this section some fundamental results required for the development of the proofs are given.

Lemma 3: ***(Positive definite matrices)*** Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a positive definite matrix and let $\mathcal{A}^{-1}$ be its inverse. Then $\mathcal{A}(i,i)\,\mathcal{A}^{-1}(i,i) \geq 1, \forall\, i \in \mathcal{V}$. Furthermore, let λ_1 be the least, λ_n the largest eigenvalue of $\mathcal{A}$, $\gamma = \lambda_n/\lambda_1$. Then $\gamma^{1/2} + \gamma^{-1/2} \geq 2 \max_{i=1,\dots,n} \left(\mathcal{A}(i,i)\mathcal{A}^{-1}(i,i)\right)^{1/2}$.

In addition, the following result on M-matrix hold [12] and [13]:

Lemma 4: ***(M-Matrix properties)*** Let $\mathcal{A}$ be an irreducible, symmetric, and strictly diagonally dominant M-matrix, then $\mathcal{A}^{-1}$ is a symmetric entrywise positive matrix and $\mathcal{A}^{-1}(i,i) > \mathcal{A}^{-1}(i,j)$, $\forall\, i,j \in \mathcal{V} : i \neq j$.

Finally, the following result concerning the inversion of the sum of two matrices holds:

Lemma 5: ***(Sherman–Morrison formula)*** Suppose $\mathcal{A}$ is an invertible square matrix and u, v are vectors. Suppose furthermore that $1 + v^T\mathcal{A}^{-1}u \neq 0$. Then

$$(\mathcal{A} + uv^T)^{-1} = \mathcal{A}^{-1} - \frac{\mathcal{A}^{-1}uv^T\mathcal{A}^{-1}}{1 + v^T\mathcal{A}^{-1}u},$$

where uv^T is the outer product of two vectors u and v.

B. Proof of Theorem 1

In order to prove the Theorem, let Q_{ij}^{-1} be the inverse of $\mathcal{Q}_{ij}$ defined in (2). The main diagonal of Q_{ij}^{-1} reads as

$$\mathcal{Q}_{ij}^{-1} = \begin{bmatrix} \left(\mathcal{Q}_i - e_h e_k^{\mathsf{T}} \mathcal{Q}_j^{-1} e_k e_h^{\mathsf{T}}\right)^{-1} & \star \\ \star & \left(\mathcal{Q}_j - e_k e_h^{\mathsf{T}} \mathcal{Q}_i^{-1} e_h e_k^{\mathsf{T}}\right)^{-1} \end{bmatrix}, \quad (8)$$

where the block off-diagonal can be neglected as they do not affect the computation of the Trace and $h \in \mathcal{V}_i$, $k \in \mathcal{V}_j$ are the node selected for the interconnections.

Let us now consider the first block on the main diagonal of the inverse matrix $\mathcal{Q}_{ij}^{-1}$. In particular, let us recall that the interconnection is obtained by connecting the h-th node of the system s_1 with the k-th node of the system s_2. Then we have $\left(\mathcal{Q}_i - e_h e_k^\mathsf{T} \mathcal{Q}_j^{-1} e_k e_h^\mathsf{T}\right)^{-1} = \mathcal{Q}_i^{-1} + \mathcal{P}_{ij}^{hk}$, where Lemma 5 has been used and $\mathcal{P}_{ij}^{hk} = \frac{\mathcal{Q}_i^{-1}(:,h)\mathcal{Q}_i^{-1}(h,:)}{1/\mathcal{Q}_j^{-1}(k,k) - \mathcal{Q}_i^{-1}(h,h)}$.

By following a similar reasoning for the second block on the main diagonal of the inverse matrix $\mathcal{Q}_{ij}^{-1}$ we obtain

$$\left(\mathcal{Q}_j - e_k e_h^\mathsf{T} \mathcal{Q}_i^{-1} e_h e_k^\mathsf{T}\right)^{-1} = \mathcal{Q}_j^{-1} + \mathcal{P}_{ij}^{kh}.$$

Since the objective is to compute the trace of the block-diagonal matrix $\mathcal{Q}_{ij}^{-1}$ given in (8), let us now investigate the structure of the eigenvalues of the two perturbations $\mathcal{P}_{ij}^{hk}$ and $\mathcal{P}_{ij}^{kh}$. In particular, by noticing that these perturbations are by construction rank-1 matrices we have that $\text{spec}(\mathcal{P}_{ij}^{hk}) = \{\lambda_{ij}^{hk}, 0, \ldots, 0\}$ and $\text{spec}(\mathcal{P}_{ij}^{kh}) = \{\lambda_{ij}^{kh}, 0, \ldots, 0\}$. In addition, the eigenvalue λ_{ij}^{hk} is by construction defined as in (3). Therefore, from the linearity of the Trace operator it follows $\text{Trace}(\mathcal{Q}_{ij}^{-1}) = \text{Trace}(\mathcal{Q}_i^{-1}) + \text{Trace}(\mathcal{Q}_j^{-1}) + \lambda_{ij}^{kh} + \lambda_{ij}^{hk}$. At this point in order to prove that $\lambda_{ij}^{hk} > 0$ and $\lambda_{ij}^{hk} > 0$ it is sufficient to show that by construction $1/\mathcal{Q}_j^{-1}(k,k) - \mathcal{Q}_i^{-1}(h,h) > 0$ and $1/\mathcal{Q}_i^{-1}(h,h) - \mathcal{Q}_j^{-1}(k,k) > 0$. In this regard, note that $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are symmetric strictly diagonally dominant M-matrices which, by construction, are irreducible being the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ associated to them (strongly) connected by definition. Then from Lemma 4 it follows that $\mathcal{Q}_i^{-1}$ and $\mathcal{Q}_j^{-1}$ are symmetric entrywise positive matrix. Furthermore, by construction we also know that $(\mathcal{Q}_r(i,i) - \sum |\mathcal{Q}_r(i,j)|) > 1, \ \forall i \in 1, \ldots, n_r$, with $r \in \{i,j\}$. Thus from [14] it follows that $\sum_{j=1}^{n_r} \mathcal{Q}_r^{-1}(i,j) \leq 1, \ \forall i \in 1, \ldots, n$, with $r \in \{i,j\}$, which in turn implies $\mathcal{Q}_i^{-1}(h,h) < 1; \mathcal{Q}_j^{-1}(k,k) < 1 \quad \forall h \in \mathcal{V}_1, \forall k \in \mathcal{V}_2$ thus the result follows.

C. Proof of Theorem 2

In order to prove the Theorem it is sufficient to show that $\lambda_{ij}^{kh} + \lambda_{ij}^{hk} > \lambda_{ij}^{k^*h^*} + \lambda_{ij}^{h^*k^*}$, for all $h \in \mathcal{V}_1$ and $k \in \mathcal{V}_2$ such that $\deg(h) < \deg(h^*)$ and $\deg(k) < \deg(k^*)$, where $\deg(h^*) = \max_{h \in \mathcal{V}_i} \deg(h)$ and $\deg(k^*) = \max_{k \in \mathcal{V}_j} \deg(k)$. In particular, it should be noticed that by construction the quantity $\lambda_{ij}^{kh} + \lambda_{ij}^{hk}$ is minimized when the terms $\mathcal{Q}_i^{-1}(h,h)$ and $\mathcal{Q}_j^{-1}(k,k)$ are *minimized* at the denominator and the terms $\|\mathcal{Q}_i^{-1}(:,h)\|^2$ and $\|\mathcal{Q}_j^{-1}(:,k)\|^2$ are *minimized* at the numerator. Therefore, the problem can be equivalently stated as proving that for all $h \in \mathcal{V}_1$ and $k \in \mathcal{V}_2$ such that $\deg(h) < \deg(h^*)$ and $\deg(k) < \deg(k^*)$, then for the system s_i we have: $\mathcal{Q}_i^{-1}(h,h) > \mathcal{Q}_i^{-1}(h^*,h^*)$ and $\|\mathcal{Q}_i^{-1}(:,h)\|^2 > \|\mathcal{Q}_i^{-1}(:,h^*)\|^2$, and a similar expression holds for the system s_j. Let us now focus on the first inequality for the system s_i as a similar reasoning will hold for the system s_j. In particular, by recalling that $\mathcal{Q}_i \, \mathcal{Q}_i^{-1} = \mathcal{I}$, for any two vertices h and h^* we have

$$\sum_{r=1}^{n_i} \mathcal{Q}_i(h^*,r)\mathcal{Q}_i^{-1}(r,h^*) = \sum_{r=1}^{n_i} \mathcal{Q}_i(h,r)\mathcal{Q}_i^{-1}(r,h) = 1. \quad (9)$$

At this point, by recalling that $Q_i(h,h) < Q_i(h^*,h^*)$ we have that

$$\mathcal{Q}_i^{-1}(h^*,h^*) = \frac{\mathcal{Q}_i(h,h)}{\mathcal{Q}_i(h^*,h^*)}\mathcal{Q}_i^{-1}(h,h) + \sum_{r=2}^{n_1} \frac{\mathcal{Q}_i(h^*,r)}{\mathcal{Q}_i(h^*,h^*)}\mathcal{Q}_i^{-1}(r,h^*) - \sum_{r=2}^{n_1} \frac{\mathcal{Q}_i(h,r)}{\mathcal{Q}_i(h^*,h^*)}\mathcal{Q}_i^{-1}(r,h),$$

Thus it follows that $\mathcal{Q}_i^{-1}(h^*,h^*) < \mathcal{Q}_i^{-1}(h,h)$ if and only if $\sum_{r=2}^{n_1} \mathcal{Q}_i(h^*,r)\mathcal{Q}_i^{-1}(r,h^*) - \sum_{r=2}^{n_1} \mathcal{Q}_i(h,r)\mathcal{Q}_i^{-1}(r,h) < (\mathcal{Q}_i(h^*,h^*) - \mathcal{Q}_i(h,h))\,\mathcal{Q}_i^{-1}(h,h)$, At this point by recalling the equality (9), the previous equation can be expressed solely in terms of the elements $\mathcal{Q}_i(h,h)$, $\mathcal{Q}_i^{-1}(h,h)$ and $\mathcal{Q}_i(h^*,h^*)$, $\mathcal{Q}_i^{-1}(h^*,h^*)$ as follows $\mathcal{Q}_i(h^*,h^*)\,\mathcal{Q}_i^{-1}(h^*,h^*) - \mathcal{Q}_i(h,h)\,\mathcal{Q}_i^{-1}(h,h) < (\mathcal{Q}_i(h^*,h^*) - \mathcal{Q}_i(h,h))\,\mathcal{Q}_i^{-1}(h,h)$. By further simplifying we have $\mathcal{Q}_i(h^*,h^*)\mathcal{Q}_i^{-1}(h^*,h^*) < \mathcal{Q}_i(h^*,h^*)\mathcal{Q}_i^{-1}(h,h)$ and thus the first inequality for the system s_i follows.

Let us now focus on the second inequality for the system s_i as again a similar reasoning will hold for the system s_j. Notice that $\|\mathcal{Q}_i^{-1}(:,h)\|^2$ represents the entry (h,h) of the matrix $\left(\mathcal{Q}_i^{-1}\right)^2$. Therefore, the result we are seeking can be obtained by following the same reasoning as before, if $\mathcal{Q}_i^2\left(\mathcal{Q}_i^{-1}\right)^2 = \mathcal{I}$ and for all $h \in \mathcal{V}_1$ such that $\deg(h) < \deg(h^*)$ we have $\mathcal{Q}_i^2(h,h) < \mathcal{Q}_i^2(h^*,h^*) \iff \mathcal{Q}_i(h,h) < \mathcal{Q}_i(h^*,h^*)$. The first property follows directly from the fact that $\mathcal{Q}_1 \, \mathcal{Q}_1^{-1} = \mathcal{I}$; while, remembering that $\mathcal{Q}_i$ is symmetric, the second property can be shown noticing that by construction the (i,i) entry of the matrix $\mathcal{Q}_1^2$ is defined as $\mathcal{Q}_1^2(h,h) = \sum_{r=1}^{n_1} \mathcal{Q}_1(h,r)\,\mathcal{Q}_1(r,h) = \|\mathcal{Q}_i(:,h)\|^2$,

D. Proof of Lemma 1

In order to prove the Lemma we must characterize an upper bound for λ_{ij}^{kh} and λ_{ij}^{hk} as defined in Theorem 1. In particular, it should be noticed that this problem can be equivalently expressed in terms of characterizing an upper bound for the terms $\mathcal{A}_i^{-1}(h,h)$, $\|\mathcal{A}_i^{-1}(:,h)\|^2$, $\mathcal{A}_j^{-1}(k,k)$, and $\|\mathcal{A}_j^{-1}(:,k)\|^2$.

Let us now focus on the two terms $\mathcal{A}_i^{-1}(h,h)$ and $\|\mathcal{A}_i^{-1}(h,:)\|^2$ as a similar reasoning will hold for the other two terms. In particular, from Lemma 3 we know that $\mathcal{A}_i^{-1}(h,h) \leq \frac{\left(\gamma_i^{1/2} + \gamma_i^{-1/2}\right)^2}{4\,\mathcal{A}_i^{\max}}$. where $\gamma_i = \lambda_{\max}(\mathcal{A}_i)/\lambda_{\min}(\mathcal{A}_i)$ and since $h = \text{argmax}_r\, \mathcal{A}_i(r,r)$ then $\mathcal{A}_i^{\max} = \mathcal{A}_i(h,h)$. At this point, by recalling that by construction $\|\mathcal{A}_i^{-1}(:,h)\|^2$ represents the entry (h,h) of the matrix $\left(\mathcal{A}_i^{-1}\right)^2$, and the eigenvalues of a squared matrix are the squared eigenvalues of the matrix itself, the following upper bound is obtained $\left\|\mathcal{A}_i^{-1}(:,h)\right\|^2 \leq \frac{\left(\gamma_i + \gamma_i^{-1}\right)^2}{4\,\mathcal{A}_i^{2\max}}$, where $\mathcal{A}_i^{2\max} = \max_r \mathcal{A}_i^2(r,r)$ and similarly to the previous case since $h = \text{argmax}_r\, \mathcal{A}_i(r,r)$ then by definition it follows that $\mathcal{A}_i^{2\max} = \mathcal{A}_i^2(h,h)$. At this point, by following the same reasoning, similar bounds can be found for the two terms $\mathcal{A}_j^{-1}(k,k)$ and $\|\mathcal{A}_j^{-1}(:,k)\|^2$. Finally, by substituting these bounds in λ_{ij}^{kh} (λ_{ij}^{hk}) and by doing simply algebraic manipulations the bounds Δ_{ij} and Δ_{ji} given in (6) follow.

E. *Proof of Lemma 2*

In order to prove the lemma, we notice that from the Gershgorin circle theorem by construction the matrix $\mathcal{Q}_{ij}$ has the following spectrum spec $(\mathcal{Q}_{ij}) \subseteq \left[1,\ \alpha + 2\ \max_{h\in\mathcal{V}_i, k\in\mathcal{V}_j}\{\deg(h), \deg(k)\} + 1\right]$.

In particular, by noticing that

$$\alpha + \max_{h\in\mathcal{V}_i, k\in\mathcal{V}_j}\{\deg(h), \deg(k)\} > \max_{h\in\mathcal{V}_i, k\in\mathcal{V}_j}\{\deg(h), \deg(k)\} + 1,$$

and by recalling that $\mathcal{Q}_r^{\max} = \alpha + \max_{p\in\mathcal{V}_r}\{\deg(p)\}, r \in i, j$ the spectrum of the matrix $\mathcal{Q}_{ij}$ can be also written as spec $(\mathcal{Q}_{ij}) \subseteq \left[1,\ 2\max\{\mathcal{Q}_i^{\max}, \mathcal{Q}_j^{\max}\}\right]$. At this stage, by recalling that $\gamma_{ij} = \lambda_{\max}(\mathcal{Q}_{ij})/\lambda_{\min}(\mathcal{Q}_{ij})$, it follows that $\gamma_{ij} < 2\max\{\mathcal{Q}_i^{\max}, \mathcal{Q}_j^{\max}\}$. Furthermore, since the matrix $\mathcal{Q}_{ij}$ in (2) is a block matrix, by construction we have $\mathcal{Q}_{ij}^{\max} = \max\left\{\mathcal{Q}_i^{\max},\ \mathcal{Q}_j^{\max}\right\}$ and ${\mathcal{Q}_{ij}^2}^{\max} = \max\left\{{\mathcal{Q}_i^2}^{\max}, {\mathcal{Q}_j^2}^{\max}\right\} + 1$.

It should be noticed that the block matrices on the main diagonal of $\mathcal{Q}_{ij}^2$ are given exactly by $\mathcal{Q}_i^2 + \mathrm{diag}(e_h)$ and $\mathcal{Q}_j^2 + \mathrm{diag}(e_k)$ with $\mathrm{diag}(e_k)$ a diagonal matrix with all zeros but the entry in the k-row and k-th column equal to 1. This follows directly from the fact that when computing $\mathcal{Q}_{ij}^2$ by construction we have $(e_h e_k^\mathsf{T})(e_k e_h^\mathsf{T}) = \mathrm{diag}(e_h)$ and $(e_k e_h^\mathsf{T})(e_h e_k^\mathsf{T}) = \mathrm{diag}(e_k)$.

F. *Proof of Theorem 3*

In order to prove the first inequality of the Theorem, let us consider for the sake of clarity a set $\mathcal{S} = \{1, 2, 3\}$ of 3 dynamical networks and assume with no lack of generality that interconnections are performed sequentially, that is first the matrix $\mathcal{Q}_1$ is interconnected with the matrix $\mathcal{Q}_2$ and then the resulting network matrix $\mathcal{Q}_{12}$ is connected with the matrix $\mathcal{Q}_3$.

At this point, by recursively applying Lemma 1, the following holds for the trace of the composite network $\mathcal{Q}_{123}$

$$\begin{aligned}\mathrm{Trace}(\mathcal{Q}_{123}^{-1}) &\leq \mathrm{Trace}(\mathcal{Q}_{12}^{-1}) + \mathrm{Trace}(\mathcal{Q}_3^{-1}) + \Delta_{12,3} + \Delta_{3,12}\\ &\leq \mathrm{Trace}(\mathcal{Q}_1^{-1}) + \mathrm{Trace}(\mathcal{Q}_2^{-1}) + \Delta_{1,2} + \Delta_{2,1}\\ &\quad + \mathrm{Trace}(\mathcal{Q}_3^{-1}) + \Delta_{12,3} + \Delta_{3,12}.\end{aligned}$$

In particular, by recalling the definition of the terms Δ_{ij} and Δ_{ji} as in (6) and by exploiting Lemma 2, the following bound for the terms $\Delta_{12,3} + \Delta_{3,12}$ is obtained with respect to the atomic parts, namely $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ $\Delta_{12,3} + \Delta_{3,12} \leq \max\{(\bar{\Delta}_{1,3} + \bar{\Delta}_{3,1}), (\bar{\Delta}_{2,3} + \bar{\Delta}_{3,2})\}$, where $\bar{\Delta}_{i,j}$ given in (7) differs from $\Delta_{i,j}$ as the γ_i and γ_j are replaced with their upper bound $2\mathcal{Q}_i^{\max}$ and $2\mathcal{Q}_j^{\max}$, respectively. Note that by using $\mathcal{Q}_i^{\max}$ and ${\mathcal{Q}_{ij}^2}^{\max}$ in $\bar{\Delta}_{i,j}$, we intrinsically exploit the equality and the lower bound given in the second and third equations of Lemma (2), respectively. Therefore we obtain

$$\begin{aligned}\mathrm{Trace}(\mathcal{Q}_{123}^{-1}) &\leq \mathrm{Trace}(\mathcal{Q}_1^{-1}) + \mathrm{Trace}(\mathcal{Q}_2^{-1}) + \mathrm{Trace}(\mathcal{Q}_3^{-1})\\ &\quad + (\Delta_{1,2} + \Delta_{2,1})\\ &\quad + \max\{(\bar{\Delta}_{1,3} + \bar{\Delta}_{3,1}), (\bar{\Delta}_{2,3} + \bar{\Delta}_{3,2})\}.\end{aligned}$$

At this point, by iterating the same reasoning for a given set $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ of n dynamical networks and by still assuming interconnections to be performed sequentially, the following bound on the trace of the composite system holds

$$\begin{aligned}\mathrm{Trace}(\mathcal{Q}^{-1}) &\leq \sum_{i=1}^{n}\left(\mathrm{Trace}(\mathcal{Q}_i^{-1}) + \max_{j=1,\ldots,i-1}\{\bar{\Delta}_{ij} + \bar{\Delta}_{ji}\}\right)\\ &\leq \sum_{i=1}^{n}\mathrm{Trace}(\mathcal{Q}_i^{-1}) + (n-1)\,\bar{\Delta}^{\max},\end{aligned}$$

where the second inequality follows from the fact that $\bar{\Delta}^{\max} = \max_{i,j=\{1,\ldots,n\}}\{\bar{\Delta}_{ij} + \bar{\Delta}_{ji}\}$ with $\bar{\Delta}_{ij} \geq \Delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ by construction. The same upper bound holds regardless of the particular sequence of interconnections and ordering of the dynamical networks. This can be explained by the fact that, by exploiting both Lemma 1 and Lemma 2, the perturbation introduced by the interconnection of any pair of (intermediate) composite networks can always be bounded from above by the max of a set of "elementary" upper bounds of the perturbation arising from network compositions involving only atomic dynamical networks, i.e., $\bar{\Delta}_{ij}$ with $i, j \in \{1, \ldots, n\}$, and for which the inequality stated above still holds true.

To prove the second inequality of the Theorem, it is sufficient to notice that by construction the perturbation terms introduced by the interconnections contribute with a positive term to the computation of the $\mathcal{H}_2$ norm of the composite network. Thus, a straightforward lower bound is given solely by the sum of the trace of the atomic parts.

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