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Structural Controllability of Symmetric Networks

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***Abstract*—The theory of structural controllability allows us to assess controllability of a network as a function of its interconnection graph and independently of the edge weights. Yet, existing structural controllability results require the weights to be selected arbitrarily and independently from one another, and provide no guarantees when these conditions are not satisfied. In this note we develop a new theory for structural controllability of networks with symmetric, thus constrained, weights. First, we show that network controllability remains a *generic* property even when the weights are symmetric. Then, we characterize necessary and sufficient graph-theoretic conditions for structural controllability of networks with symmetric weights: a symmetric network is structurally controllable if and only if it is structurally controllable without weight constraints. Finally, we use our results to assess structural controllability from one region of a class of empirically-reconstructed brain networks.**

***Index Terms*—Network controllability; structural controllability; interconnected systems; graph theory; symmetric networks.**

I. Introduction

The question of controllability of complex network systems arising in engineering, social, and biological domains has been the subject of intensive study in the last few years [1]–[3]. One key question motivating the investigation is to characterize relationships and tradeoffs between the interconnection structure of a network and its controllability [4]–[6]. To this end, graphical tools from structural systems theory [7]–[10] are typically preferred over algebraic controllability tests, which suffer from numerical instabilities when the network cardinality grows, require exact knowledge of the network weights, and are agnostic to the graph supporting the dynamics.

While the theory of structured systems and generic properties of linear systems is well-developed and understood [11], all results assume the possibility of assigning the network weights arbitrarily and independently from one another. In fact, when this condition is violated, the conclusions drawn from structural analysis may lead to incorrect results [7], [12]. Unfortunately, it is often the case that this assumption is violated in real networks due to physical, technological, or biological reasons. For instance, the small-signal network-preserving model of a power network contains a Laplacian submatrix, whose entries are symmetric and satisfy linear constraints (row sums equal to zero) [13], [14]. Similar constraints appear also when studying synchronization in networks of Kuramoto oscillators [15] and general systems with consensus dynamics [16]. Novel theories and tools are needed to study controllability of networks with constrained weights.

In this paper we focus on networks with symmetric weights and derive graph-theoretic conditions for their structural controllability from dedicated control inputs. While (group) symmetry has previously been found to be responsible for network uncontrollability [17], [18], the question of how symmetric edge weights affect structural controllability has not been investigated, with the exception of [19]. In [19], however, the proposed conditions for structural controllability of undirected (symmetric) networks are implicit and based on the generalized zero forcing sets to estimate the dimension of the controllable subspace. Similarly, although the recent paper [20] studies structural controllability for a class of networks with constrained parameters, this class of network matrices does not contain the set of symmetric matrices considered in this work. Thus, the necessary and sufficient conditions derived in this paper are the first graph-theoretic conditions for structural controllability of networks with symmetric weights.

The contribution of this paper is three-fold. First, we show that controllability of symmetric networks is a generic property. That is, either the network is controllable for almost all symmetric choices of interconnection weights, or it is not controllable for all symmetric weights. This first result can be easily extended to different classes of constraints other than symmetry. Second, we show that a network with symmetric weights is structurally controllable if and only if it is spanned by a (symmetric) cactus rooted at the control node. By comparing our result with those in [21], our analysis shows that a network is structurally controllable with symmetric weights if and only if it is structurally controllable with unconstrained weights. Third, we use our results to show that a class of (symmetric) brain networks reconstructed from diffusion MRI data is structurally controllable from a single dedicated control region. Finally, we note that, due to duality between controllability and observability, the results of this paper extend directly to the study of structural observability of networks with symmetric weights and a dedicated sensor.

The rest of the paper is organized as follows. Section II contains our network model and preliminary notions. Section III contains our analysis and conditions for structural controllability of networks with symmetric weights, and some examples. Finally, Section IV contains an illustrative example featuring brain networks, and Section V concludes the paper.

II. Problem setup and preliminary notions

We study controllability of symmetric network systems, which are described by a weighted directed graph (digraph)

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$\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively, and a symmetric weighted adjacency matrix $A = [a_{ij}]$ with $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$ and $a_{ij} \in \mathbb{R}$ if $(i,j) \in \mathcal{E}$. Let $x \in \mathbb{R}^n$ be the vector containing the state of the network nodes over time, and let $i \in \mathcal{V}$ be the control node. We let x evolve according to linear time-invariant dynamics:

$$\delta(x) = Ax + b^{\mathrm{i}} u, \tag{1}$$

where $\delta(x)$ denotes the time derivative (resp. time shift) operator for continuous-time (resp. discrete-time) dynamics, and $b^{\mathrm{i}} = e_i$, with e_i the i-th canonical vector of dimension n. Finally, let the controllability matrix of (1) be

$$\mathcal{C}(A, b^{\mathrm{i}}) = \begin{bmatrix} b^{\mathrm{i}} & Ab^{\mathrm{i}} & \cdots & A^{n-1}b^{\mathrm{i}} \end{bmatrix}, \tag{2}$$

and recall that the network (1) is controllable if and only if its controllability matrix $\mathcal{C}(A, b^{\mathrm{i}})$ is invertible [22].

Assessing controllability of network systems is numerically difficult because the controllability matrix typically becomes ill-conditioned as the network cardinality increases; e.g., see [4], [23]. Because different controllability tests suffer similar numerical difficulties, a convenient tool to study controllability of networks is to resort to the theory of structural systems. To formalize this discussion, let $a_{\mathcal{E}} = \{a_{ij} \; : \; (i,j) \in \mathcal{E}\}_{\text{ordered}}$ denote the set of nonzero entries of A in lexicographic order, and notice that the determinant $\det(\mathcal{C}(A, b^{\mathrm{i}})) = \phi(a_{\mathcal{E}})$ is a polynomial function with variables $a_{\mathcal{E}}$. From the above reasoning, the network (1) is uncontrollable when the weights $a_{\mathcal{E}}$ are chosen so that $\phi(a_{\mathcal{E}}) = 0$. Let $\mathcal{S}$ contain the choices of weights that render the network (1) uncontrollable; that is,

$$\mathcal{S} = \{z \in \mathbb{R}^d \; : \; \phi(z_1, \dots, z_d) = 0\}, \tag{3}$$

where $d = |\mathcal{E}| = |a_{\mathcal{E}}|$. Notice that $\mathcal{S}$ describes an algebraic variety of $\mathbb{R}^d$ [24]. This implies that controllability of (1) is a *generic property*, as it fails to hold on an algebraic variety of the parameter space [24]–[26]. Consequently, when assessing controllability of the network (1) as a function of the weights, only two mutually exclusive cases are possible:

(i) either there is *no* choice of weights a_{ij}, with $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$, rendering the network (1) controllable; or

(ii) the network (1) is controllable for all choices of weights a_{ij}, with $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$, except, possibly, those belonging to the proper algebraic variety $\mathcal{S} \subset \mathbb{R}^d$.[1]

Loosely speaking, if one can find a choice of weights such that the network (1) is controllable, then *almost all* choices of weights yield a controllable network. In this case, the network is said to be *structurally controllable* [7], [21], [27].

Classical results on structural controllability cannot be directly applied to networks where the weights are constrained [7], [12]. In fact, these results assume that the network weights can be selected arbitrarily and independently from one another, a condition that cannot be satisfied, for instance, when the weights need to be symmetric. In this note we overcome this limitation, and extend the results on structural controllability to symmetric networks. In particular we show that a network is structurally controllable with symmetric weights if and only if it is structurally controllable with unconstrained weights.

[1]The variety $\mathcal{S}$ of $\mathbb{R}^d$ is proper when $\mathcal{S} \neq \mathbb{R}^d$ [24].

III. STRUCTURAL CONTROLLABILITY OF SYMMETRIC NETWORKS

In this section we derive necessary and sufficient graph-theoretic conditions for structural controllability of networks with symmetric weights. We proceed as follows. First, we show that network controllability remains a generic property when the weights are symmetric. Second, we provide conditions to construct controllable networks with symmetric weights. Finally, combining these results yields conditions for structural controllability of networks with symmetric weights.

Theorem 3.1: ***(Symmetry and genericity)*** Controllability of the network (1) with symmetric matrix A is a generic property.

Proof: Let $d = |\mathcal{E}|$ and $d_s = |\{i \; : \; (i,i) \in \mathcal{E}\}|$. Notice that a network with symmetric weights is uniquely specified by $(d + d_s)/2$ parameters, for instance, by the set $a'_{\mathcal{E}} = \{a_{ij} \; : \; (i,j) \in \mathcal{E}, \; i \leq j\}_{\text{ordered}}$ in lexicographic order. Further, because of the symmetry constraint, the determinant of the controllability matrix of (1) is a polynomial function $\phi'(a'_{\mathcal{E}})$, which can be obtained, for instance, from the determinant $\det(\mathcal{C}(A, b^{\mathrm{i}}))$ by substituting a_{ij} with a_{ji} whenever $i > j$. Thus, even for symmetric networks, the determinant of the controllability matrix is a polynomial function of the network weights, and the weights that render the network uncontrollable define the algebraic variety $\mathcal{P} = \{z \in \mathbb{R}^{(d+d_s)/2} \; : \; \phi'(z_1, \dots, z_{(d+d_s)/2}) = 0\}$. To conclude, either $\mathcal{P} = \mathbb{R}^{(d+d_s)/2}$, and the network is uncontrollable for all choices of symmetric weights, or $\mathcal{P}$ is a proper algebraic variety of $\mathbb{R}^{(d+d_s)/2}$, and the network is controllable for all choices of symmetric weights except, if any, those belonging to the set $\mathcal{P}$ of zero Lebesgue measure [24]. ■

Theorem 3.1 shows that controllability remains a generic property even when the weights are constrained to be symmetric. This result will be key in the derivation of our conditions for structural controllability of networks with symmetric weights. In fact, because controllability remains a generic property, it will be sufficient to show that a network is controllable for a specific choice of symmetric weights to guarantee that controllability holds for almost all choices of weights. In the next example we illustrate that the set of symmetric weights preventing controllability forms an algebraic variety.

Example 1: ***(Structural controllability with symmetric weights)*** Consider a network with symmetric adjacency matrix

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}, \tag{4}$$

and input vector $b^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T}$. From (2), the controllability matrix of the pair (A, b^1) is

$$\mathcal{C}(A, b^1) = \begin{bmatrix} 1 & 0 & a_{12}^2 + a_{13}^2 \\ 0 & a_{12} & a_{13}a_{23} \\ 0 & a_{13} & a_{12}a_{23} \end{bmatrix}, \tag{5}$$

with determinant $\det(\mathcal{C}(A, b^1)) = a_{23}a_{12}^2 - a_{23}a_{13}^2$. Thus, the network is controllable (i.e., $\det(\mathcal{C}(A, b^1)) \neq 0$) for all

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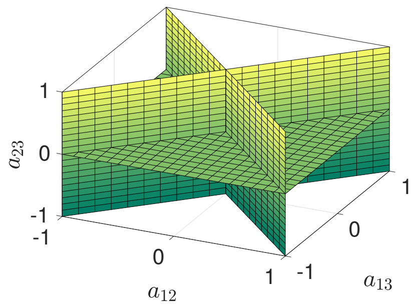

Fig. 1. Algebraic variety defined by $a_{23}a_{12}^2 - a_{23}a_{13}^2 = 0$, which determines the weights for which the network in Example 1 is not controllable. The network is controllable for all weights outside of this algebraic variety.

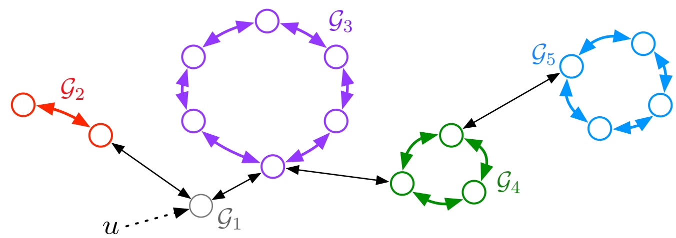

Fig. 2. A sym-cactus $\mathcal{G} = \overline{\bigcup}_{i=1}^5 \mathcal{G}_i$ rooted at the control node. Sym-cycles $\mathcal{G}_2, \dots, \mathcal{G}_5$ are highlighted with different colors. Notice that $\mathcal{G}_1$ is not a sym-cycle because it comprises 1 node without a self-loop. See Definition 2.

symmetric choices of weights a_{12}, a_{13}, and a_{23}, except those lying on the proper algebraic variety shown in Fig. 1 and defined by the equation $a_{23}a_{12}^2 - a_{23}a_{13}^2 = 0$. $\square$

*Remark 1: **(Structural controllability of consensus systems)*** A multi-agent consensus network with leader nodes is described by a linear time-invariant dynamical system, where the nonzero entries of A have a specified sign and the sums along the rows of A equal to a constant (1 for discrete-time networks, and 0 in the case of continuous-time networks) [28], [29]. Theorem 3.1 can be easily extended to include constraints on the sign of the entries of A and on their sums. In fact, if $\sum_{j=1}^n a_{ij} = c$, for some constant $c \in \mathbb{R}$, then[2] $a_{i1} = c - \sum_{j=2}^n a_{ij}$ can be substituted in the polynomial $\det(\mathcal{C}(A, b^{\mathrm{i}}))$, showing that the set of parameters preventing controllability forms an algebraic variety of the free parameter space, and that controllability remains a generic property despite the constraints. Similarly, when some entries have a specified sign or need to assume identical values, the set of parameters preventing controllability can be shown to be a subset of an algebraic variety, which either equals the set of feasible parameters, or remains of zero Lebesgue measure. $\square$

*Example 2: **(Structural controllability of consensus systems)*** Consider a linear discrete-time consensus system with node 1 as a leader and adjacency matrix

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

Because the rows of A need to sum to 1, it is possible to rewrite 3 parameters as a function of the others. For instance, rewrite $a_{13} = 1 - a_{12}$, $a_{23} = 1 - a_{21}$, and $a_{32} = 1 - a_{31}$. By doing so, the determinant $\det(\mathcal{C}(A, b^1)) = -a_{21}^2 a_{31} + a_{21}^2 + a_{21}a_{31}^2 - a_{31}^2$, and the set of weights that make such determinant vanish defines a proper algebraic variety of the parameter space $\mathbb{R}^3$. $\square$

We next introduce some graph-theoretic notions [7], [30]. Given a digraph, a path is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is a directed edge of the digraph. A digraph is strongly connected if there exists a directed path from any node to any other node. Furthermore, given the digraphs $\mathcal{G}_1, \dots, \mathcal{G}_m$, let $\mathcal{G} = \overline{\bigcup}_{i=1}^m \mathcal{G}_i$ be the connected digraph $(\mathcal{V}, \mathcal{E})$ defined as follows: $\mathcal{V} = \bigcup_{i=1}^m \mathcal{V}_i$ and $\mathcal{E} = \bigcup_{i=1}^m \mathcal{E}_i \cup \bar{\mathcal{E}}$, where $|\bar{\mathcal{E}}| = 2(m-1)$ and, for all $i \in \{2, \dots, m\}$, there is a unique pair of edges $(p_i, q_i) \in \bar{\mathcal{E}}$ and $(q_i, p_i) \in \bar{\mathcal{E}}$ with $p_i \in \mathcal{V}_i$ and $q_i \in \bigcup_{j=1}^{i-1} \mathcal{V}_j$. Finally, we present some definitions that are inspired by [21] and will be used to derive our structural controllability conditions for networks with symmetric weights.

[2]If $a_{i1} = 0$, then select a different nonzero entry.

*Definition 1: **(Sym-cycle)*** A sym-cycle is a strongly connected digraph with $n \geq 1$ nodes, edge set $\{(i,j) : |i-j| = 1\} \cup \{(1,n),(n,1)\}\}$, and symmetric weights $a_{ij} = a_{ji}$. $\square$

From Definition 1, the adjacency matrix of a sym-cycle is

$$A = \begin{cases} a_{ij} \neq 0 & \text{if } |i-j| = 1 \text{ or } (i,j) \in \{(1,n),(n,1)\}, \\ a_{ij} = 0 & \text{otherwise.} \end{cases} \tag{6}$$

*Definition 2: **(Sym-cactus)*** A sym-cactus is a strongly connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined as $\mathcal{G} = \overline{\bigcup}_{i=1}^m \mathcal{G}_i$ and satisfying the following properties:

(i) $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is a sym-cycle if $|\mathcal{V}_1| > 1$ (if $|\mathcal{V}_1| = 1$, we allow $\mathcal{G}_1$ to contain no edges, that is, $\mathcal{E}_1 = \emptyset$),
(ii) $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ is a sym-cycle for every $i \in \{2, \dots, m\}$,
(iii) the node sets satisfy $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$, whenever $i \neq j$. $\square$

Notice that, if $|\mathcal{V}_1| = 1$, the graph $\mathcal{G}_1$ in Definition 2 can either be a sym-cycle (thus having a self-loop), or a node without self-loop.

*Remark 2: **(Stem, buds, cactus, and sym-cactus)*** Our definitions of sym-cycle and sym-cactus are compatible with the classic notions of stem, bud, and cactus as defined in [21]. In particular, because we focus on networks with symmetric weights, stems, buds, and cacti [21] become equivalent to interconnected sym-cycles. $\square$

We say that a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is spanned by $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ if $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. Further, the sym-cactus $\mathcal{G} = \overline{\bigcup}_{i=1}^m \mathcal{G}_i$ is rooted at the node i if i is a node of $\mathcal{G}_1$. Fig. 2 illustrates the definitions of sym-cycle and sym-cactus rooted at node i.

The following lemma shows that every sym-cycle is structurally controllable from any node. That is, for almost all symmetric choices of network weights, every cycle network is controllable independently of the location of the control node.

*Lemma 3.2: **(Every sym-cycle is structurally controllable)*** Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a sym-cycle. The pair (A, b^{i}) is structurally controllable for all $i \in \{1, \dots, n\}$.

Proof: Owing to Theorem 3.1 we need to show that, for every sym-cycle and control node, there exists a choice of weights rendering the network controllable. Without affecting

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generality, we assume that the control node is $i = 1$ (if $i \neq 1$, simply apply a similarity transformation PAP^T via a permutation matrix P to reorder the nodes as desired).

If $n \leq 2$, the network is clearly controllable. For $n > 2$, partition the matrix A as

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{12} \in \mathbb{R}^{1\times(n-1)}$, $A_{21} \in \mathbb{R}^{(n-1)\times 1}$ and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$. Notice that A_{22} is a tridiagonal matrix.

Suppose that the pair (A, b^1) is not structurally controllable. Then, for all choices of weights, there exists an eigenvector[3] v of A such that $v^\mathsf{T} b^1 = 0$ [22]. Thus, $v = [v_1, v_2, \dots, v_n]^\mathsf{T} = [v_1, \bar{v}]^\mathsf{T} = [0, \bar{v}]^\mathsf{T}$, and the eigenproblem $Av = \lambda v$ becomes

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda\bar{v} \end{bmatrix}. \tag{7}$$

From (7), the pair (A, b^1) is uncontrollable if and only if A_{22} has an eigenvector $\bar{v}$ that lies in the null space of A_{12}. Equivalently, $A_{22}\bar{v} = \lambda\bar{v}$ and $A_{12}\bar{v} = a_{12}\bar{v}_1 + a_{1n}\bar{v}_{n-1} = 0$.

Assign all the weights of A_{22} as 1 (or any other constant), and notice that A_{22} is a Toeplitz tridiagonal matrix with eigenvectors $\bar{v}^i = [\bar{v}^i_j] = \left[\sin\left(\frac{ji\pi}{n}\right)\right]$, for $i, j \in \{1, \dots, n-1\}$ [31, Example 7.2.5]. Finally, to ensure controllability of (A, b^1), select a_{12} and a_{1n} such that, for all $i \in \{1, \dots, n-1\}$,

$$a_{12}\sin\left(\frac{i\pi}{n}\right) + a_{1n}\sin\left(\frac{(n-1)i\pi}{n}\right) \neq 0. \tag{8}$$

Notice that (8) can be ensured by $|a_{12}| \neq |a_{1n}|$. In fact,

(i) for i odd, $\sin\left(\frac{i\pi}{n}\right) = \sin\left(\frac{(n-1)i\pi}{n}\right)$ because

$$\pi - \frac{i\pi}{n} + 2k\pi - \frac{(n-1)i\pi}{n} = (2k+1-i)\pi = 0,$$

by selecting $k = (i-1)/2$;

(ii) for i even, $\sin\left(\frac{i\pi}{n}\right) = -\sin\left(\frac{(n-1)i\pi}{n}\right)$ because

$$\frac{(n-1)i\pi}{n} - 2k\pi + \frac{i\pi}{n} = (i-2k)\pi = 0,$$

by selecting $k = i/2$.

This concludes the proof. ∎

Lemma 3.2 implies that every sym-cycle is controllable for almost every symmetric choice of weights. In particular, the following choice of weights yields a controllable sym-cycle from node i (see the proof of Lemma 3.2 and Eq. (8)):

$$A = \begin{cases} a_{ij} = a, & \text{if } |i-j| = 1, \text{ and} \\ a_{1n} = a_{n1} = b, & \text{with } |a| \neq |b|, \end{cases} \tag{9}$$

for some nonzero constants a and b. We next show that sym-cacti are also a fundamentally controllable structure contained in every structurally controllable symmetric network.

Theorem 3.3: ***(Structural controllability of symmetric networks with one control node)*** The network $\mathcal{G}$ with control node i is structurally controllable with symmetric weights if and only if it is spanned by a sym-cactus rooted at i.

The proof of Theorem 3.3 is postponed to the Appendix. In Theorem 3.3 we show that a necessary and sufficient condition for structural controllability of networks with symmetric weights is the existence of a spanning sym-cactus rooted at the control node. This result implies that the symmetry constraint on the network weights does not prevent controllability if the same unconstrained network is structurally controllable. It should be noticed that a sym-cactus is not, in general, strongly structurally controllable [32]. That is, there exist choices of weights that render a sym-cactus uncontrollable. We next illustrate a systematic procedure to construct an uncontrollable sym-cactus composed of controllable sym-cycles.

Example 3: ***(Uncontrollable sym-cactus)*** Consider the sym-cactus $\mathcal{G} = \mathcal{G}_1 \overline{\cup} \mathcal{G}_2$ with control node 1 and adjacency matrix

$$A = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & c_2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & c_2 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & -4 & 3 & 0 \end{array}\right],$$

where the diagonal blocks are the adjacency matrices of the sym-cycles $\mathcal{G}_1$ and $\mathcal{G}_2$, and the remaining blocks denote the interconnection between $\mathcal{G}_1$ and $\mathcal{G}_2$ with weight c_2. It can be verified that $\lambda_0 = -2$ is a transmission zero of the system [22]

$$\begin{aligned} \delta(x) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ y &= \begin{bmatrix} 1 & 2 \end{bmatrix} x, \end{aligned}$$

and that the pair (A, b^1) is uncontrollable when c_2 satisfies

$$\left(\begin{bmatrix} 2 & 3 & -4 \\ 3 & 2 & 3 \\ -4 & 3 & 2 \end{bmatrix}^{-1}_{1,1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}_{2,2} \right)^{-\frac{1}{2}} = 6.292853089,$$

where $[M]^{-1}_{i,i}$ denotes the i-th diagonal entry of the matrix M^{-1}. The reader is referred to the proof of Theorem 3.3, *Case 2.a*, for a detailed derivation of this result. □

Following the above discussion and the derivation in the proof of Theorem 3.3, we next describe an algorithm to assign the weights of a sym-cactus to guarantee controllability. To this aim, let $\mathrm{spec}(M)$ denote the spectrum of the matrix M, and notice that the adjacency matrix A of the sym-cactus mbox $\mathcal{G} = \overline{\bigcup}_{i=1}^m \mathcal{G}_i$ can be written recursively as $(k = 2, \dots, m)$

$$A_k = \begin{bmatrix} A_{k-1} & c_k e_{q_k} e_1^\mathsf{T} \\ c_k e_1 e_{q_k}^\mathsf{T} & H_k \end{bmatrix}, \tag{10}$$

where H_k is the adjacency matrix of $\mathcal{G}_k$, $A_1 = H_1$, $A_m = A$, and $c_k \neq 0$, for some index $q_k \in \{1, \dots, \sum_{j=1}^{k-1} |\mathcal{V}_j|\}$.[4] Let

$$A_{k-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{11}$$

where A_{11} is a scalar, and let $\mathcal{Z}_k = \{\lambda : A_{12}(A_{22} - \lambda I)^{-1} e_{q_k} = 0\}$ be the zeros of the single-input single-output system $(A_{22}, e_{q_k}, A_{12})$. Then, the pair (A, b^1) can be made

[3]Since $A = A^\mathsf{T}$, we do not distinguish between left and right eigenvectors.

[4]This recursive construction follows directly from Definition 2.

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Algorithm 1: *Design of controllable sym-cactus*

Input : $\{H_k \;:\; k = 1, \ldots, m,\; H_k$ satisfying (6)$\}$;
Output : Controllable pair (A, b^1), with A adjacency matrix of the sym-cactus $\mathcal{G} = \overline{\bigcup}_{i=1}^{m} \mathcal{G}_i$ rooted at 1;

Select the weights of H_1 as in (9);
Set $A_1 = H_1$;
for $k = 2 : m$ **do**
  Partition A_{k-1} according to (11) ;
  Select $q_k \in \{1, \ldots, \sum_{j=1}^{k-1} |\mathcal{V}_j|\}$;
  Select the weights of H_k as in (9) and so that $\mathrm{spec}(H_k) \cap \mathrm{spec}(A_{22}) = \emptyset$;
  Compute $\mathcal{Z} = \{\lambda \;:\; A_{12}(A_{22} - \lambda I)^{-1} e_{q_k} = 0\}$;
  Select $c_k \neq c_\lambda$ for every $\lambda \in \mathcal{Z}$, where $c_\lambda = \left(\left[H_k - \lambda I \right]_{1,1}^{-1} \left[A_{22} - \lambda I \right]_{q_k, q_k}^{-1} \right)^{-\frac{1}{2}}$;
  Generate A_k as in (10);
return $A = A_m$. The pair (A, b^1) is controllable;

controllable by selecting the weights in $\mathcal{G}_k$ to recursively satisfy the following conditions:

(i) (H_k, b^1) is controllable (see (9) for a choice of weights),
(ii) $\mathrm{spec}(H_k) \cap \mathrm{spec}(A_{22}) = \emptyset$, and
(iii) for all $\lambda \in \mathcal{Z}_k$, $c_k^{-2} \neq \left[H_k - \lambda I \right]_{1,1}^{-1} \left[A_{22} - \lambda I \right]_{q_k, q_k}^{-1}$,

A procedure to construct a controllable sym-cactus is summarized in Algorithm 1, whose complexity is linear in the number of sym-cycles and cubic in their dimension.

Remark 3: ***(Structural controllability of symmetric networks with multiple dedicated control nodes)*** Theorem 3.3 can be extended to the case of multiple dedicated control nodes; that is, when the input matrix in (1) satisfies $B = \begin{bmatrix} e_{c_1} & \cdots & e_{c_m} \end{bmatrix}$ and $\{c_1, \ldots, c_m\} \subseteq \mathcal{V}$ is the set of control nodes. In particular, a network $\mathcal{G}$ with m control nodes $\{c_1, \ldots, c_m\}$ is structurally controllable with symmetric weights if and only if it is spanned by a disjoint union of sym-cacti rooted at the nodes $\{c_1, \ldots, c_m\}$. The necessity of this result follows directly from [33, Theorem 1], while its sufficiency is obtained by applying the same steps as in the proof of Theorem 3.3 to each disjoint cactus. □

We conclude this section with an example of structural controllability in the case of multiple dedicated control nodes.

Example 4: ***(Structural controllability with symmetric weights and multiple dedicated control nodes)*** Consider the network in Fig. 3(a) with adjacency matrix

$$A = \left[\begin{array}{c|c c c|c c c} 0 & 0 & a_{13} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{23} & a_{24} & 0 & 0 & 0 \\ a_{13} & a_{23} & 0 & a_{34} & a_{35} & 0 & 0 \\ 0 & a_{24} & a_{34} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{35} & 0 & 0 & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & a_{56} & 0 & a_{67} \\ 0 & 0 & 0 & 0 & a_{57} & a_{67} & 0 \end{array}\right]$$

and control vector $b^1 = e_1$. The pair (A, b^1) is structurally controllable because of Theorem 3.3. In fact, there exists a sym-cactus $\mathcal{G} = \overline{\bigcup}_{i=1}^{3} \mathcal{G}_i$ that spans the network and is rooted at 1. Consider now the network in Fig. 3(c) with adjacency

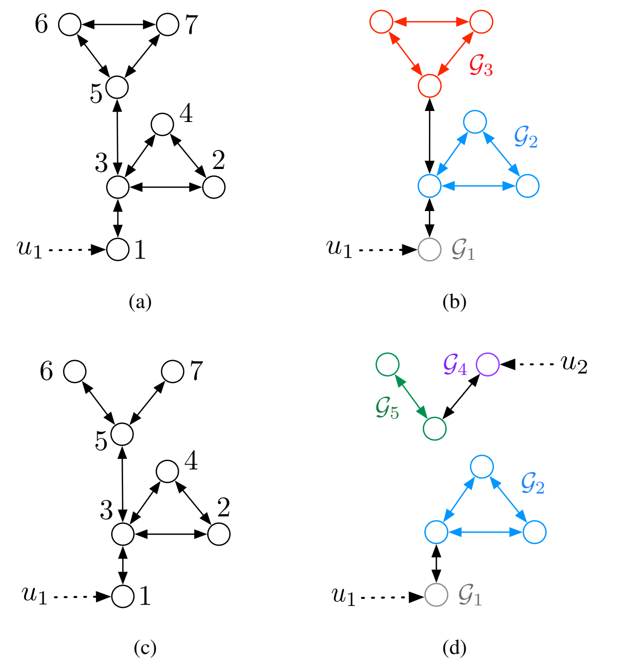

Fig. 3. The networks considered in Example 4. (a) The network of the pair (A, b^1). (b) The network is structurally controllable because it is spanned by a sym-cactus rooted at 1. (c) The network of the pair $(\tilde{A}, b^1)$ is not structurally controllable with only one control input at node 1. (d) By adding a control input at node 7, the network recovers structural controllability because it is spanned by a disjoint union of sym-cacti rooted at nodes 1 and 7, respectively.

matrix $\tilde{A} = A$ and disconnect nodes 6 and 7; that is, $a_{67} = 0$. The pair $(\tilde{A}, b^1)$ is not structurally controllable because there is no sym-cactus that spans the network and is rooted at 1. However, by connecting an additional input at node 7, it is possible to span the network with a disjoint union of sym-cacti. That is, there exist distinct sym-cacti $\mathcal{G}_1 \overline{\cup} \mathcal{G}_2$ and $\mathcal{G}_4 \overline{\cup} \mathcal{G}_5$ that span the network and are rooted at 1 and 7, respectively. Therefore, by setting $b^2 = e_7$, the pair $(\tilde{A}, [b^1\; b^2])$ is structurally controllable with symmetric weights. □

IV. Application to Structural Brain Networks

We apply our analysis to a class of structural brain networks reconstructed from diffusion magnetic resonance imaging (MRI) data,[5] where nodes correspond to well-known brain regions and edges correspond to white matter connections between them [35]. The network dynamics can be derived from the linearization of a general noise-free Wilson-Cowan system [36] and read as $\delta(x) = Ax + b^{\mathrm{i}} u$, where A is a symmetric matrix that represents the anatomical connectivity of the brain. Further, $u : \mathbb{N} \to \mathbb{R}$ is the control input applied to the i-th brain region, and $x : \mathbb{N} \to \mathbb{R}^n$ is the vector containing the state of the brain regions over time. Examples of state values range from the magnitude of electrical activity [37] to the quantity of oxyhemoglobin and deoxyhemoglobin in the hemodynamic response [38]. Although brain dynamics

[5]Diffusion magnetic resonance images were acquired for a total of eight subjects in triplicate (mean age 27 ± 5 years, two female, two left handed), and at each scanning session a $T1$-weighted anatomical scan was acquired. For each subject, $n = 234$ regions were registered as areas of interest [34].

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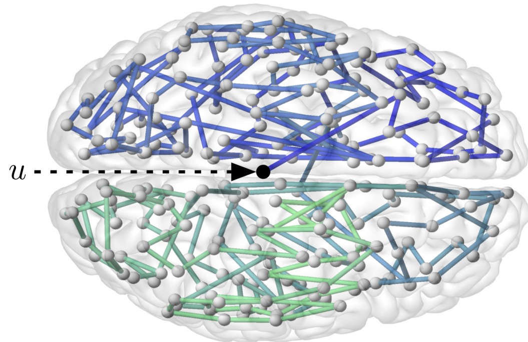

Fig. 4. Axial view of the structural brain network and a spanning Hamiltonian path. Each node represents a brain region of the anatomical scans. The Hamiltonian path starts from the region representing the control node (*brain stem*). Regions are plotted according to the mean location of voxels in each of the 234 parcels in the Lausanne atlas [35] and averaged over the cohort of healthy adult subjects. This figure was obtained with *BrainNet Viewer* [42].

may be nonlinear at the micro-scale, the study of linear network models for macro-scale neural dynamics has been validated in several studies (see e.g. [39]), and has given access to theoretical and practical tools that are particularly useful around an operating point [34], [40]. Controllability of this class of networks has been examined in different studies, including [34], via numerical controllability tests. Yet, because of the large cardinality of these networks, most controllability tests suffer from numerical instabilities, sometimes leading to competing conclusions [34], [41]. Further, because typical diffusion MRI techniques produce symmetric adjacency matrices, the graphical investigation of structural controllability for this type of networks was, up to now, not possible.

As illustrated in Fig. 4, the brain networks in our dataset are spanned by a Hamiltonian path,[6] which is a special case of a sym-cactus, starting from the control node. Theorem 3.3 implies that, despite having symmetric weights, networks reconstructed from diffusion MRI data are structurally controllable from a single brain region, thus controllable for almost every symmetric choice of weights.

V. Conclusion

In this note we derive necessary and sufficient graph-theoretic conditions for structural controllability of networks with symmetric weights and one control node. Because weights need to be symmetric, classic results from structural systems theory cannot be directly applied. Surprisingly, we show that network controllability remains a generic property even when the weights are symmetric, and that a network with symmetric weights is structurally controllable if and only if its unconstrained equivalent network is structurally controllable; that is, if and only if it is spanned by a (symmetric) cactus.

While our analysis focuses on symmetric weights and a single control node, as discussed in Remark 1 and 3, our results extend directly to other classes of parameter constraints and to the case of multiple dedicated control nodes. The case of non-dedicated control nodes, however, requires different definitions and reasoning, and it is left as the subject of future research.

[6]A path in a graph is Hamiltonian if it visits all the vertices exactly once.

Appendix

We now prove some instrumental results and Theorem 3.3.

Lemma A.1: (***Controllability of subsystems***) Consider the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with control nodes $\mathcal{K} \subset \mathcal{V}$, input matrix $B_{\mathcal{K}} = \begin{bmatrix} e_1 & \dots & e_m \end{bmatrix}$, and adjacency matrix A partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{m \times m}$ and $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$. If the pair $(A, B_{\mathcal{K}})$ is controllable, then (A_{22}, A_{21}) is also controllable.

Proof: If (A_{22}, A_{21}) is not controllable, then there exists an eigenvector v_2 associated with $\lambda \in \operatorname{spec}(A_{22})$ satisfying [22]

$$v_2^\mathsf{T} \begin{bmatrix} A_{21} & A_{22} - \lambda I \end{bmatrix} = 0.$$

Let $v^\mathsf{T} = \begin{bmatrix} 0^\mathsf{T} & v_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$, and notice that

$$\begin{bmatrix} 0^\mathsf{T} & v_2^\mathsf{T} \end{bmatrix} \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & A_{22} - \lambda I \end{bmatrix} = 0.$$

Then, v is a left eigenvector of A associated with the eigenvalue $\lambda \in \operatorname{spec}(A)$, and it satisfies $v^\mathsf{T} B_{\mathcal{K}} = 0$. This implies that $(A, B_{\mathcal{K}})$ is not controllable, and concludes the proof. ■

Lemma A.2: (***Eigenspace of perturbed matrix***) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and let $\Delta = e_i e_i^\mathsf{T}$, with $i \in \{1, \dots, n\}$. Then, $\lambda \in \operatorname{spec}(A + c\Delta)$ for all $c \in \mathbb{R}$ if and only if there exists $v \neq 0$ satisfying $(A - \lambda I)v = 0$ and $\Delta v = 0$.

Proof: *(If)* The sufficiency of the statement follows by noting that $(A - \lambda I + c\Delta)v = (A - \lambda I)v = 0$. *(Only if)* Let the vectors $v_c \neq 0$ and $v_0 \neq 0$ satisfy $v_c^\mathsf{T}(A - \lambda I + c\Delta) = 0$ and $(A - \lambda I)v_0 = 0$, respectively. Then, for all $c \in \mathbb{R}$, $v_c^\mathsf{T}(A - \lambda I + c\Delta)v_0 = c v_c^\mathsf{T} \Delta v_0 = 0$. Let $v_{\bar{c}}$ denote the vector v_c with $c = \bar{c} \neq 0$. Notice that, because $\Delta = e_i e_i^\mathsf{T}$, $v_{\bar{c},i} v_{0,i} = 0$, where $v_{\bar{c},i}$ and $v_{0,i}$ denote the i-th element of $v_{\bar{c}}$ and v_0, respectively. Let $v = v_0$ if $v_{0,i} = 0$, and $v = v_{\bar{c}}$ otherwise. To conclude, notice that $v \neq 0$, $\Delta v = 0$, and $(A - \lambda I)v = 0$. ■

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3: *(Only if)* Assume that $\mathcal{G}$ is structurally controllable from the node i. From [21], there must exist a directed cactus $\mathcal{D}$ rooted at i that spans $\mathcal{G}$. Because $\mathcal{G}$ has symmetric weights, this also implies the existence of a sym-cactus, which is obtained by adding edges to $\mathcal{D}$ to make it symmetric. See Remark 2 for a discussion of directed and symmetric cacti.

(If) Let the network be spanned by the sym-cactus $\mathcal{G} = \overline{\bigcup}_{i=1}^{m} \mathcal{G}_i$ rooted at the control node. Let A_k be the adjacency matrix of $\overline{\bigcup}_{i=1}^{k} \mathcal{G}_i$, $k \leq m$. Without loss of generality, we assume $b^{\mathrm{i}} = b^1$ (if $b^{\mathrm{i}} \neq b^1$, reorder the nodes). We will construct a controllable realization (A_m, b^1), thus proving that the original network admits a controllable realization. The claimed statement then follows from Theorem 3.1.

We proceed by induction. In the base step, Lemma 3.2 concludes on the controllability of the pair (A_1, b^1). In the inductive step, we assume that (A_{k-1}, b^1) is controllable, and show that (A_k, b^1) is controllable. Let A_k be partitioned as

$$A_k = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{12}$$

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where $A_{11} \in \mathbb{R}$, $A_{22} \in \mathbb{R}^{(n_1-1)\times(n_1-1)}$, and $A_{33} \in \mathbb{R}^{n_2 \times n_2}$, with n_1 and n_2 being the dimension of A_{k-1} and the difference between the dimension of A_k and A_{k-1}, respectively. Notice that A_{33} corresponds to H_k in decomposition (10). We show that A_k has no eigenvector v of the form

$$v = \begin{bmatrix} 0 & v_1^\mathsf{T} & v_2^\mathsf{T} \end{bmatrix}^\mathsf{T}, \tag{13}$$

which, by the eigenvector test, implies that (A_k, b^1) is controllable. Due to the definition of the operator $\bigcup$ (a single connection between adjacent sym-cycles) and by exploiting the decomposition of A_k in (10), we have that either

(1) $A_{32} = A_{23}^\mathsf{T} = 0$ and $A_{31} = A_{13}^\mathsf{T} = c_k e_1 \neq 0$, or
(2) $A_{31} = A_{13}^\mathsf{T} = 0$ and $A_{32} = A_{23}^\mathsf{T} = c_k e_1 e_{q_k}^\mathsf{T} \neq 0$,

where e_1, e_{q_k} are canonical vectors of appropriate dimensions.

Case (1) Consider the eigenproblem $A_k v = \lambda v$. For v to be of the form (13), λ must be an eigenvalue of both A_{22} and A_{33}. Therefore, by choosing the weights in A_{33} such that $\mathrm{spec}(A_{33}) \cap \mathrm{spec}(A_{22}) = \emptyset$, we obtain a controllable (A_k, b^1). Notice that such a choice of weights always exists because A_{33} has generically full rank.[7] For instance, given a full rank realization of A_{33}, we can multiply A_{33} by a suitable constant $c \in \mathbb{R}$ to guarantee that $\mathrm{spec}(cA_{33}) \cap \mathrm{spec}(A_{22}) = \emptyset$.

Case (2) Define the matrix $P(\lambda)$ as

$$P(\lambda) = \begin{bmatrix} A_{12} & 0 \\ A_{22} - \lambda I & A_{23} \\ A_{32} & A_{33} - \lambda I \end{bmatrix}.$$

Due to (13), the eigenproblem $A_k v = \lambda v$ reduces to

$$P(\lambda) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \tag{14}$$

We will show that $P(\lambda)$ is full rank for all λ, thus ensuring that an eigenvector as (13) cannot exist. As in *Case (1)*, we choose weights in A_{33} such that $\mathrm{spec}(A_{33}) \cap \mathrm{spec}(A_{22}) = \emptyset$. Thus, we consider 3 cases:

(2.a) $\lambda \notin \mathrm{spec}(A_{22}) \cup \mathrm{spec}(A_{33})$,
(2.b) $\lambda \in \mathrm{spec}(A_{22})$, and
(2.c) $\lambda \in \mathrm{spec}(A_{33})$.

Case (2.a) Because $A_{22} - \lambda I$ and $A_{33} - \lambda I$ are invertible,

$$\begin{aligned} &\mathrm{Rank}\,(P(\lambda)) \\ &= \mathrm{Rank}\left(P(\lambda) \begin{bmatrix} (A_{22} - \lambda I)^{-1} & 0 \\ 0 & (A_{33} - \lambda I)^{-1} \end{bmatrix} \right) \\ &= \mathrm{Rank}\left(\begin{bmatrix} A_{12}(A_{22} - \lambda I)^{-1} & 0 \\ I & c_k T_3 \\ c_k T_2 & I \end{bmatrix} \right) \end{aligned}$$

where $T_2 = e_1 e_{q_k}^\mathsf{T} (A_{22} - \lambda I)^{-1}$, and $T_3 = e_{q_k} e_1^\mathsf{T} (A_{33} - \lambda I)^{-1}$.

Notice that, for any vector v_3 of appropriate dimension we have $T_3 v_3 = \alpha e_{q_k}$, for some value α dependent on λ and A_{33}. Similarly, $T_2 v_2 = \beta e_1$, for some value β dependent on λ and A_{22}. Further, for any fixed λ, there exists a value c_k such that

$$\begin{bmatrix} I & c_k T_3 \\ c_k T_2 & I \end{bmatrix} \tag{15}$$

is invertible. In fact, elementary column operations reveal that

$$\mathrm{Rank}\left(\begin{bmatrix} I & c_k T_3 \\ c_k T_2 & I \end{bmatrix} \right) = \mathrm{Rank}\left(\begin{bmatrix} I - c_k^2 T_3 T_2 & c_k T_3 \\ 0 & I \end{bmatrix} \right).$$

Notice that $T_3 T_2$ is a rank-1 matrix and that $\mathrm{spec}(I - T_3 T_2) = \{1, \ldots, 1, 1 - c_k^2 \tilde{\lambda}\}$, where $\tilde{\lambda}$ is the only nonzero eigenvalue of $T_3 T_2$. Thus, (15) is invertible whenever $c_k^2 \neq \tilde{\lambda}^{-1}$.

Let $\mathcal{Z} = \{\lambda \;:\; A_{12}(A_{22} - \lambda I)^{-1} e_{q_k} = 0\}$, and let c_k be such that (15) is invertible for all $\lambda \in \mathcal{Z}$. Then, $P(\lambda)$ is also full rank for all $\lambda \in \mathcal{Z}$. Next, assume by contradiction that $P(\lambda)$ loses rank for some value $\bar{\lambda} \notin \mathcal{Z}$. Then, there exist nonzero w_2 and w_3 such that

$$\begin{bmatrix} A_{12}(A_{22} - \bar{\lambda} I)^{-1} & 0 \\ I & c_k T_3 \\ c_k T_2 & I \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = 0.$$

We have $w_2 = -c_k T_3 w_3 = -c_k \alpha e_{q_k}$, and $c_k \alpha A_{12}(A_{22} - \bar{\lambda} I)^{-1} e_{q_k} = 0$. Notice that $\alpha \neq 0$. Otherwise, $T_3 w_3 = 0$ and, consequently, $w_2 = 0$ and $w_3 = 0$. Further, $A_{12}(A_{22} - \bar{\lambda} I)^{-1} e_{q_k} \neq 0$ because $\bar{\lambda} \notin \mathcal{Z}$. We conclude that, when c_k is such that (15) is invertible for all $\lambda \in \mathcal{Z}$, $P(\lambda)$ is full rank.

Case (2.b) Because $A_{33} - \lambda I$ is invertible,

$$\begin{aligned} \mathrm{Rank}\,(P(\lambda)) &= \mathrm{Rank}\left(P(\lambda) \begin{bmatrix} I & 0 \\ 0 & (A_{33} - \lambda I)^{-1} \end{bmatrix} \right) \\ &= \mathrm{Rank}\left(\begin{bmatrix} A_{12} & 0 \\ A_{22} - \lambda I & c_k T_3 \\ A_{32} & I \end{bmatrix} \right), \end{aligned}$$

where $T_3 = e_{q_k} e_1^\mathsf{T} (A_{33} - \lambda I)^{-1}$. By means of elementary column operations we obtain

$$\mathrm{Rank}\,(P(\lambda)) = \mathrm{Rank}\left(\begin{bmatrix} A_{12} & 0 \\ A_{22} - \lambda I - c_k T_3 A_{32} & c_k T_3 \\ 0 & I \end{bmatrix} \right).$$

Notice that, if $\lambda \notin \mathrm{spec}(A_{22} - c_k T_3 A_{32})$ for some c_k, then $P(\lambda)$ can be made full rank by a selection of c_k. Instead, if $\lambda \in \mathrm{spec}(A_{22} - \lambda I - c_k T_3 A_{32})$ for all values of c_k, then, due to Lemma A.2, $(A_{22} - \lambda I - c_k T_3 A_{32}) v = 0$, for some fixed eigenvector v and for all c_k. Because (A_k, b^1) is controllable by the induction hypothesis, so is the pair $(A_{22}, A_{21}) = (A_{22}, A_{12}^\mathsf{T})$ by Lemma A.1. We conclude that $A_{12} v \neq 0$. This implies that, for all values of c_k, the submatrix

$$\begin{bmatrix} A_{12} \\ A_{22} - \lambda I - c_k T_3 A_{32} \end{bmatrix}$$

is full rank and, consequently, so is $P(\lambda)$.

Case (2.c) Because $A_{22} - \lambda I$ is invertible,

$$\begin{aligned} &\mathrm{Rank}\,(P(\lambda)) \\ &= \mathrm{Rank}\left(P(\lambda) \begin{bmatrix} (A_{22} - \lambda I)^{-1} & 0 \\ 0 & I \end{bmatrix} \right) \\ &= \mathrm{Rank}\left(\begin{bmatrix} A_{12}(A_{22} - \lambda I)^{-1} & 0 \\ I & A_{23} \\ c_k T_2 & A_{33} - \lambda I \end{bmatrix} \right), \end{aligned}$$

where $T_2 = e_1 e_{q_k}^\mathsf{T} (A_{22} - \lambda I)^{-1}$. By means of elementary row operations we obtain that $\mathrm{Rank}\,(P(\lambda))$ equals

$$\mathrm{Rank}\left(\begin{bmatrix} A_{12}(A_{22} - \lambda I)^{-1} & 0 \\ I & A_{23} \\ 0 & A_{33} - \lambda I - c_k A_{23} T_2 \end{bmatrix} \right).$$

[7] The graph with adjacency matrix A_{33} contains a set of n_2 edges, for instance $\mathcal{M} = \{(1,2), (2,3), \ldots, (n_2 - 1, n_2), (n_2, 1)\})$, where no two edges point to the same node. Such set of edges is called a matching of size n_2, and its existence guarantees that A_{33} is generically full rank [43, §1.1.2].

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Notice that, if $\lambda \notin \text{spec}(A_{33} - c_k A_{23} T_2)$ for some c_k, then $P(\lambda)$ can be made full rank by a selection of c_k. Instead, if $\lambda \in \text{spec}(A_{33} - \lambda I - c_k A_{23} T_2)$ for all values of c_k, then, due to Lemma A.2, $(A_{33} - \lambda I - c_k A_{23} T_2)v = 0$, for some fixed eigenvector v and for all c_k. Because (A_{33}, b^{i}) can be made controllable for all indices i due to Lemma 3.2, the submatrix

$$\begin{bmatrix} A_{23} \\ A_{33} - \lambda I - c_k A_{23} T_2 \end{bmatrix} \tag{16}$$

is full rank. To make $P(\lambda)$ full rank, we proceed by contradiction. Suppose there exist nonzero v_1 and v_2 such that

$$\begin{bmatrix} A_{12}(A_{22} - \lambda I)^{-1} & 0 \\ I & A_{23} \\ 0 & A_{33} - \lambda I - c_k A_{23} T_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Notice that $v_1 = -A_{23} v_2 = -c_k e_{q_k} e_1^{\mathsf{T}} v_2$ and that v_1 has exactly one nonzero entry (q_k) due to (16) being full rank. Finally, $A_{12}(A_{22} - \lambda I)^{-1} v_1 = 0$ implies that λ must be a transmission zero of the single-input single-output system $(A_{22}, e_{q_k}, A_{12})$ [22]. Thus, $P(\lambda)$ can be made full rank by selecting A_{33} such that its eigenvalues are different from the transmission zeros of the system $(A_{22}, e_{q_k}, A_{12})$.

In conclusion, by choosing A_{33} and the interconnection weight c_k as discussed in *Cases (1)*, *(2.a)*, *(2.b)*, and *(2.c)*, we obtain a controllable realization of the sym-cactus $\mathcal{G} = \bigcup_{i=1}^{m} \mathcal{G}_i$, thus concluding the inductive procedure. ■

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