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The Structured Controllability Radius of Symmetric (Brain) Networks

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Abstract— **In this paper we propose and analyze a novel notion of controllability of network systems with linear dynamics and symmetric weights. Namely, we quantify the controllability degree of a network with its distance from the set of uncontrollable networks with the same structure, that is, with the minimum Frobenius norm of a structured perturbation rendering the network uncontrollable (structured controllability radius). We derive analytical conditions to compute the structured controllability radius of a network with symmetric weights, and illustrate our results through a number of examples. In particular, we use our theoretical results to study the controllability properties of a set of brain networks reconstructed from diffusion MRI data, and compare them with the controllability properties of a class of random networks. Our results show that brain networks feature a controllability radius that is consistently smaller than the one of random networks with similar weights, indicating that the considered brain networks may not be optimized to favor controllability.**

I. Introduction

The question of controllability of natural and man-made network systems has recently received considerable attention. In the context of the human brain, the study of various controllability properties may not only shed light into the organization and function of different neural circuits, but also inform the design and implementation of minimally invasive yet effective intervention protocols to treat neurological disorders [1]. Although the study of the human brain as a network system is still in its infancy, some recent results, e.g., see [2], [3], [4], have suggested that the complexity of the brain and its underlying principles can be further untangled with tools from control theory and network science [5].

While the dynamics of most brain processes is clearly nonlinear, linearized models with empirically reconstructed network matrices have been proved useful to characterize how the anatomical structure of the brain influences its dynamic functions [6], [7]. In this paper we follow this line of work, and model the dynamics of a brain network as a linear, discrete-time, time-invariant system, where the network matrix is empirically estimated from diffusion MRI data. A key feature of these empirically reconstructed networks is that the estimated edges are undirected, giving rise to symmetric network matrices [8]. This constraint on the edge weights adds a layer of complexity to the study of network controllability [9], [10]. For instance, it prevents the use of most tools developed within structural control theory [11].

In this paper we propose and analyze a novel notion of controllability for symmetric networks, namely, the structured controllability radius. Specifically, we quantify the controllability degree of a network with the size of the smallest symmetric perturbation (measured with the Frobenius norm) that has a given sparsity pattern and renders the network uncontrollable. We provide analytical conditions to compute the structured controllability radius of a symmetric network, and use these conditions to compare a set of brain networks with a class of random networks. Our results show that the considered brain networks feature a controllability radius that is consistently smaller when compared to the considered random networks, suggesting that the topological organization of the brain may lead to unique dynamical features different from those of random network models [12].

Related work Different notions of controllability of a system have been proposed over the years. Starting from the binary definition of controllability proposed in [13], Gramian-based metrics have been proposed to provide a quantitative measure of the controllability degree of a system and, more recently, of a network based on the energetic effort needed to control the state towards a desired value [14], [15]. In [16], [17] an alternative notion of controllability is introduced, where the controllability degree of a system is quantified by the smallest norm of a perturbation of the system parameters causing uncontrollability. Later, this notion of controllability radius has been extended to account for several types of constrained perturbations (Hermitian, symmetric, and skew-symmetric) [18]. Yet, with the exception of [19], the use of the controllability radius to quantify the controllability degree of a network has not been investigated. In this case, because only existing edges can typically be modified, perturbations need to feature a pre-specified sparsity pattern, a constraint that renders classic results on the controllability radius inapplicable. In this paper we improve upon existing results, particularly [19], by focusing on symmetric and structured perturbations, by deriving an explicit set of equations for the computation of the structured controllability radius, and by exploiting our results to compare a class of brain networks with random networks with similar weights.

Contribution The contribution of this paper is three-fold. First, we propose a novel notion of controllability degree for networks with symmetric adjacency matrix, namely, the structured controllability radius, which equals the smallest Frobenius norm of a symmetric perturbation that renders the network uncontrollable and has a pre-specified sparsity pattern. Second, we derive explicit necessary and sufficient

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conditions for the computation of the structured controllability radius, and illustrate our procedure through various examples. Third and finally, we use our notion of structured controllability radius to compare a class of brain networks reconstructed from diffusion MRI data with a set of random networks with similar weights and topologies. Our results show that the controllability radius of brain networks is consistently smaller than in the case of random networks, suggesting that the anatomical organization of the brain may favor dynamic properties different from controllability.

Paper organization The remainder part of the paper is organized as follows. In Section II we introduce our network model, we define different controllability metrics, and we state the controllability radius optimization problem. In Section III we derive our conditions for the computation of the structured controllability radius. Section IV contains our numerical study of the structured controllability radius of brain and random networks. Section V concludes the paper.

Mathematical notation $\text{supp}(\cdot)$ denotes the support of a vector and $\text{vec}(\cdot)$ denotes the vectorization of a matrix. $\lambda_{\min}(\cdot)$ and $\sigma_{\min}(\cdot)$ denote the minimum eigenvalue and singular value of a matrix. $\circ$ and $\otimes$ denote the Hadamard (element-wise) and Kronecker products, respectively. $\mathbb{1}_n \in \mathbb{R}^n$ ($\mathbb{1}_{n\times n} \in \mathbb{R}^{n\times n}$) denotes a vector (matrix) of all ones. $(\cdot)^+$ denotes the Moore-Penrose pseudo inverse of a matrix. $\|\cdot\|_{\text{F}}$, $\|\cdot\|_2$ and $tr(\cdot)$ denote the Frobenius norm, spectral norm and trace of a matrix, respectively. e_i denotes the i-th canonical vector. Finally, we denote a positive definite (positive semi-definite) matrix A with $A > 0$ ($A \geq 0$).

II. Model and problem statement

We consider networks represented by a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the node and edge sets, respectively. Let $A = [a_{ij}]$ be the weighted adjacency matrix of $\mathcal{G}$, where $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$ and $a_{ij} \in \mathbb{R}$ if $(i,j) \in \mathcal{E}$. Because we study brain networks reconstructed from diffusion MRI images [8], we assume that $A = A^\mathsf{T}$. An example of adjacency matrix is in Fig. 1.

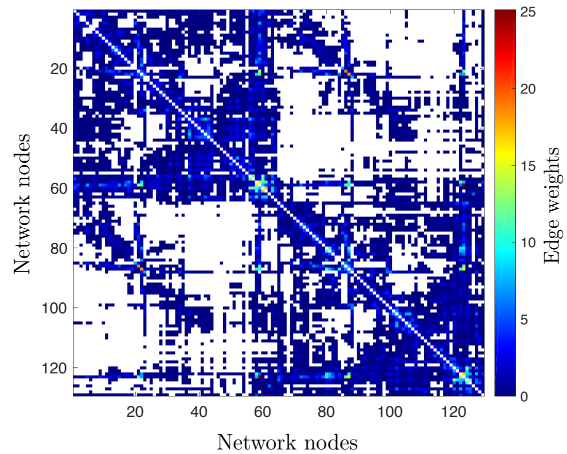

Fig. 1. Example of an anatomical connectivity matrix derived from diffusion MRI scans of a brain. Notice that entries are symmetric and the diagonal entries are zero.

The network dynamics is described by the following discrete-time linear time-invariant system:

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ is the vector containing the state of the nodes over time, $u : \mathbb{N} \to \mathbb{R}^m$ is the control input that is applied to the network through the input matrix $B \in \mathbb{R}^{n\times m}$. Without loss of generality, we assume that B has full rank.

The system (1) is controllable if there exists a control input that can steer the system from a given initial state to any desired final state. Several notions exist to quantify the controllability degree of a system. One such metric measures the control energy required to control the state between two values, and is quantified by the controllability Gramian

$$W = \sum_{\tau=0}^{\infty} A^\tau B B^\mathsf{T} (A^\mathsf{T})^\tau.$$

Notice that $W \geq 0$ and, further, $W > 0$ if and only if the pair (A, B) is controllable [20]. As a known result in system theory [14], [15], the upper bound for the control energy required to steer the network to a desired final state is inversely proportional to $\lambda_{\min}(W)$. Thus, if the eigenvalues of W are large, the required control energy is small and hence, the system has a larger controllability degree. In the case of brain networks, a larger controllability degree might enable or guide the use of less invasive treatments.

An alternative characterization of the degree of controllability of a system is provided by the controllability radius, which is defined as the smallest norm of a perturbation that renders the system uncontrollable. Mathematically, such perturbation is obtained by solving the minimization problem

$$\begin{aligned} \mu = \min_{\Delta_A, \Delta_B} \quad & \|[\Delta_A \ \Delta_B]\|_2 \\ \text{s.t.} \quad & (A + \Delta_A, B + \Delta_B) \text{ is uncontrollable.} \end{aligned}$$

Equivalently [21],

$$\mu = \min_{s\in\mathbb{C}} \sigma_{\min}\left([sI - A, \ B]\right).$$

Clearly, $\mu > 0$ if and only if $W > 0$. Further, when μ is small, a small perturbation of the system weights exist that renders the system uncontrollable.

Although the above metrics $\lambda_{\min}(W)$ and μ provide useful information regarding controllability degree of a system, they may not be directly applicable to networks due to the following reasons. First, the definition of μ allows for only *unstructured* perturbations Δ_A, Δ_B. In contrast, feasible network perturbations may be subject to constraints on their sparsity patterns and edge weights, as in the case of the networks considered in this paper. Second, the above metrics are in terms of smallest eigenvalue/singular value and, consequently, they do not provide any insight about the magnitude and the distribution of the individual entries of the perturbation. For instance, they do not help in determining which edges of a network are more (or less) sensitive to perturbations with respect to making the system uncontrollable.

To overcome these limitations and to make the metric μ meaningful for structured networks, we reformulate the optimization problem to include both symmetry and sparsity constraints explicitly in the problem. We use the Frobenius norm to measure the size of the perturbation. Further, we consider perturbations only on the network weights, that is, $\Delta_B = 0$. For simplicity, in the remainder of the paper we denote Δ_A by Δ. We introduce the sparsity constraints on Δ via a constraint graph $\mathcal{H} = (\mathcal{V}, \mathcal{E}_{\mathcal{H}})$, where $\mathcal{E}_{\mathcal{H}}$ denotes the set of edges that can be perturbed.[1] Let H be the 0-1 adjacency matrix associated with $\mathcal{H}$, and let $H^c = \mathbb{1}_{n\times n} - H$ be the unweighted complimentary adjacency matrix. Then, the sparsity constraints can be written as $H^c \circ \Delta = 0$. Since Δ is symmetric, we also assume H to be symmetric.[2]

The structured controllability radius of the pair (A, B) is the solution of the following minimization problem:

$$\min_{\Delta, v, \lambda} \quad \|\Delta\|_{\mathrm{F}}^2 \tag{2}$$

$$\begin{aligned}
\text{s.t.} \quad & \Delta = \Delta^{\mathsf{T}}, && \text{(symmetry constraint)} && \text{(2a)}\\
& (A+\Delta)v = \lambda v, && \text{(eigenvalue constraint)} && \text{(2b)}\\
& \|v\|_2^2 = 1, && \text{(eigenvector constraint)} && \text{(2c)}\\
& v^{\mathsf{T}} B = 0, && \text{(uncontrollability)} && \text{(2d)}\\
& H^c \circ \Delta = 0, && \text{(structural constraint)} && \text{(2e)}
\end{aligned}$$

where constraint (2d) follows from the PBH uncontrollability test [20], and constraint (2c) is for uniqueness of v. Notice that the minimization problem (2) is not convex due to the eigenvalue constraint (2b). Consequently, multiple local minima may exist. This is a common feature in various minimum distance and eigenvalue assignment problems [22]. We conclude this section with the following remark.

Remark 1: ***(Structured vs unstructured controllability radius)*** The minimization problem (2) admits the trivial solution $\Delta = 0$ if and only if the pair (A, B) is uncontrollable. Further, the controllability radius without structural constraints is always finite, that is, a finite perturbation causing uncontrollability always exists. Instead, the sparsity constraints (2e) may render the problem unfeasible (trivially, in the case where (A, B) is controllable and $H = 0$). □

III. SOLUTION TO THE OPTIMIZATION PROBLEM

In this section we derive a solution to the non-convex optimization problem (2). In the theory of equality constrained non-linear programming, the first-order optimality conditions are meaningful only when the optimal points satisfy the regularity condition given by $\operatorname{Rank} J = n_c$, where J is the Jacobian of the constraints and n_c equals the total number of independent equality constraints. This regularity condition is mild and usually satisfied for most classes of problems [23]. Before presenting the main result, we derive the Jacobian and state the regularity condition for the optimization problem (2). Given the constraint graph $\mathcal{H}$, let n_{s} and $\bar{n}_{\mathrm{s}}$ satisfy

$$\begin{aligned}
n_{\mathrm{s}} &= |\{(i,j) \ : \ H^c = [h_{ij}],\, j \geq i,\, h_{ij} = 1\}|,\\
\bar{n}_{\mathrm{s}} &= |\{(i,j) \ : \ H^c = [h_{ij}],\, h_{ij} = 1\}|,
\end{aligned}$$

and note that the constraint (2e) can be equivalently written as (see the proof of Lemma 3.1 for a formal definition of Q)

$$Q\mathrm{vec}(\Delta) = 0, \tag{3}$$

for some 0-1 matrix Q of dimension $\bar{n}_{\mathrm{s}} \times n^2$.

Lemma 3.1: ***(Jacobian of the constraints)*** The Jacobian of the equality constraints (2a)-(2e) is given by

$$J(\Delta, v, \lambda) = \begin{bmatrix} I - T_n & 0 & 0\\ v^{\mathsf{T}} \otimes I & A + \Delta - \lambda I & -v\\ 0 & 2v^{\mathsf{T}} & 0\\ 0 & B^{\mathsf{T}} & 0\\ Q & 0 & 0 \end{bmatrix}, \tag{4}$$

where T_n is the n^2-dimensional permutation matrix satisfying $\mathrm{vec}(\Delta^{\mathsf{T}}) = T_n \mathrm{vec}(\Delta)$, and Q is as in (3). Further, the total number of independent scalar constraints in (2a)-(2e) is

$$n_c = \frac{n^2+n}{2} + n + 1 + m + n_s. \tag{5}$$

Proof: We construct the Jacobian $J(\Delta, v, \lambda)$ by rewriting the constraints (2a)-(2e) in vectorized form and taking their derivatives with respect to $\delta \triangleq \mathrm{vec}(\Delta)$, v and λ. Vectorization of (2a) yields $(I - T_n)\delta = 0$ and its derivatives read as the first block row of J. Among the total n^2 scalar constraints in (2a), $\frac{n^2-n}{2}$ are redundant resulting in only $\frac{n^2+n}{2}$ independent constraints. Using the property $\mathrm{vec}(AB) = (B^{\mathsf{T}} \otimes I)\mathrm{vec}(A)$, re-vectorization of (2b) yields $(A - \lambda I)v + (v^{\mathsf{T}} \otimes I)\delta = 0$, from which we obtain the second block row of J. Notice that (2b) consists of n scalar constraints. Differentiation of (2c) and (2d) is straightforward and it provides 1 and m rows, respectively. Finally, (2e) consists of $\bar{n}_s$ non-trivial sparsity constraints, which can be written as $Q\delta = 0$ where $Q = [e_{i_1} \ e_{i_2} \ \ldots \ e_{i_{\bar{n}_s}}]^{\mathsf{T}}$ and $\{i_1, \ldots, i_{\bar{n}_s}\} = \mathrm{supp}(\mathrm{vec}(H^c))$ is the set of indices indicating the ones in $\mathrm{vec}(H^c)$. Because H^c is symmetric, only n_s constraints of (2e) in the lower-triangular (or upper-triangular) part of H^c are independent when we combine (2e) with the independent constraints of (2a). Thus, the total number of independent constraints in (2a)-(2e) is $n_c = \frac{n^2+n}{2} + n + 1 + m + n_s$ and this concludes the proof. ■

We now solve the minimization problem (2).

Theorem 3.2: ***(Structured controllability radius of symmetric networks)*** Let Δ^*, v^* and λ^* satisfy the constraints (2a)-(2e). Then, Δ^* is a local minimum of the minimization problem (2) if and only if, for some $l^* \in \mathbb{R}^n$ and $q^* \in \mathbb{R}^m$,

$$\Delta^* = -\frac{1}{4} H \circ \left[v^*(l^*)^{\mathsf{T}} + l^*(v^*)^{\mathsf{T}} \right], \tag{6a}$$

$$\begin{bmatrix} A + \Delta^* - \lambda^* I & B \end{bmatrix} \begin{bmatrix} l^* \\ q^* \end{bmatrix} = 0, \tag{6b}$$

$$(v^*)^{\mathsf{T}} l^* = 0, \tag{6c}$$

$$\operatorname{Rank} J(\Delta^*, v^*, \lambda^*) = n_c, \text{ and} \tag{6d}$$

$$P^* D^* P^* \geq 0, \tag{6e}$$

[1] The graph $\mathcal{H}$ has the same nodes as $\mathcal{G}$, but possibly different edge set.

[2] If H is not symmetric, construct a symmetric H' by removing a minimal set of edges from $\mathcal{H}$. It can be shown that the optimization problems with constraints H and H', respectively, admit the same solutions.

where n_c is as in (5), D^* is the Hessian defined as

$$D^* = \begin{bmatrix} 2I & I \otimes l^* & 0 \\ I \otimes (l^*)^\mathsf{T} & 0 & -l^* \\ 0 & -(l^*)^\mathsf{T} & 0 \end{bmatrix},$$

P^* is the projection matrix of $J(\Delta, v, \lambda)$ defined as

$$P^* = I - J^+(\Delta^*, v^*, \lambda^*) J(\Delta^*, v^*, \lambda^*).$$

Proof: We prove the result using the Lagrange theorem for equality constrained minimization [24]. Let $S \in \mathbb{R}^{n\times n}$, $l \in \mathbb{R}^n$, $h \in \mathbb{R}$, $q \in \mathbb{R}^m$, and $M \in \mathbb{R}^{n\times n}$ be the Lagrange multipliers associated with constraints in (2), respectively. We make use of the following properties for the proof:

(i) $tr(A) = tr(A^\mathsf{T})$ and $tr(AB) = tr(BA)$,
(ii) $\|A\|_\text{F}^2 = tr(A^\mathsf{T}A) = \text{vec}^\mathsf{T}(A)\text{vec}(A)$,
(iii) $\mathbb{1}_n^\mathsf{T}(A \circ B)\mathbb{1}_n = tr(A^\mathsf{T}B)$,
(iv) $A \circ B = B \circ A$ and $A \circ (B \circ C) = (A \circ B) \circ C$,
(v) $A \circ (B + C) = (A \circ B) + (A \circ C)$ and $(A \circ B)^\mathsf{T} = A^\mathsf{T} \circ B^\mathsf{T}$,
(vi) $\frac{d}{dX} tr(X^\mathsf{T}X) = 2X$ and $\frac{d}{dX} tr(AX) = A^\mathsf{T}$,
(vii) $a^\mathsf{T}Xy = y^\mathsf{T}(I \otimes a^\mathsf{T})\text{vec}(X) = \text{vec}^\mathsf{T}(X)(I \otimes a)y$,

The Lagrange function for the optimization problem is

$$\begin{aligned}
&\mathcal{L}(\Delta, v, \lambda, S, l, h, q, M) = \|\Delta\|_\text{F}^2 + \mathbb{1}_n^\mathsf{T}[S \circ (\Delta - \Delta^\mathsf{T})]\mathbb{1}_n \\
&+ l^\mathsf{T}(A + \Delta - \lambda I)v + h(\|v\|_2^2 - 1) + q^\mathsf{T}B^\mathsf{T}v \\
&+ \mathbb{1}_n^\mathsf{T}[M \circ (H^c \circ \Delta)]\mathbb{1}_n = \\
&\overset{(a)}{=} tr(\Delta^\mathsf{T}\Delta) + tr[(S^\mathsf{T} - S)\Delta] + l^\mathsf{T}(A + \Delta - \lambda I)v \\
&+ h\left(v^\mathsf{T}v - 1\right) + q^\mathsf{T}B^\mathsf{T}v + tr[(M \circ H^c)^\mathsf{T}\Delta]
\end{aligned}$$

where (a) follows from properties (i)-(iv). Next, we derive the first-order necessary conditions for a local minimum. Differentiating $\mathcal{L}$ w.r.t. Δ and equating to 0, we get

$$\frac{d}{d\Delta}\mathcal{L} \overset{\text{(vi)}}{=} 2\Delta + S - S^\mathsf{T} + lv^\mathsf{T} + M \circ H^c = 0. \quad (7)$$

Taking the Hadamard product of (7) with H^c and using $H^c \circ \Delta = 0$ and $H^c \circ H^c = H^c$, we get

$$(S - S^\mathsf{T}) \circ H^c + (lv^\mathsf{T}) \circ H^c + M \circ H^c = 0. \quad (8)$$

Replacing $M \circ H^c$ from (8) into (7), we get

$$\Delta = -\frac{1}{2} H \circ (S - S^\mathsf{T} + lv^T). \quad (9)$$

Since H is symmetric, the transpose of (9) yields

$$\Delta = \Delta^\mathsf{T} \overset{\text{(v)}}{=} -\frac{1}{2} H \circ (S^\mathsf{T} - S + vl^\mathsf{T}). \quad (10)$$

Adding (9) and (10) and using (v), we obtain (6a).

Next, we differentiate $\mathcal{L}$ w.r.t. v and equate to 0:

$$(A + \Delta - \lambda I)^\mathsf{T} l + 2hv + Bq = 0. \quad (11)$$

Pre-multiplying (11) by v^T and using the eigenvalue, eigenvector and uncontrollability constraints, we get $h = 0$. Then, since A and Δ are symmetric, (11) yields (6b).

Finally, differentiating $\mathcal{L}$ w.r.t. λ and equating to 0, we get the orthogonality constraint (6c).

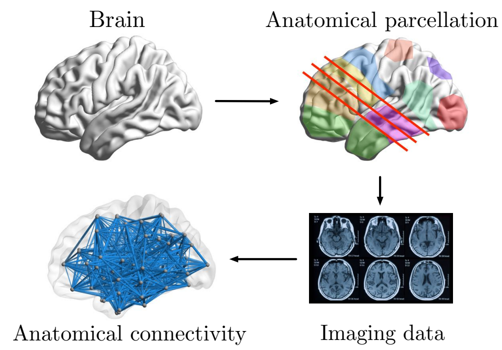

Fig. 2. Main steps performed to obtain anatomical connectivity matrices.

Equation (6d) is the necessary regularity condition and follows from Lemma 3.1 Next, we obtain the second-order sufficient conditions by deriving the Hessian of $\mathcal{L}$. Recall that $\delta = \text{vec}(\Delta)$. Taking the differential of $\mathcal{L}$ twice, we get

$$\begin{aligned}
d^2\mathcal{L} &= 2tr((d\Delta)^\mathsf{T}d\Delta) + 2h(dv)^\mathsf{T}dv + 2l^\mathsf{T}(d\Delta - Id\lambda)dv \\
&\overset{(b)}{=} 2(d\delta)^\mathsf{T}d\delta + 2(dv)^\mathsf{T}(I \otimes l^\mathsf{T})d\delta - 2d\lambda l^\mathsf{T}dv \\
&= [(d\delta)^\mathsf{T}, (dv)^\mathsf{T}, d\lambda]\; D\; [(d\delta)^\mathsf{T}, (dv)^\mathsf{T}, d\lambda]^\mathsf{T},
\end{aligned}$$

where (b) follows from properties (ii), (vii) and $h = 0$. The sufficient second-order optimality condition for the optimization problem requires the Hessian matrix to be positive semidefinite in the kernel of the Jacobian at the optimal point [25]. That is, $z^\mathsf{T}D^*z \geq 0$, $\forall z : J(\Delta^*, v^*, \lambda^*)z = 0$. This condition is equivalent to $P^*D^*P^* \geq 0$, since $J(\Delta^*, v^*, \lambda^*)z = 0$ if and only if $z = P^*u$ for any $u \in \mathbb{R}^{n^2+n+1}$ [23]. Since the projection matrix P^* is symmetric, (6e) follows and this concludes the proof. ■

Remark 2: ***(Computing an optimal solution)*** Observe that Δ^* in (6a) is symmetric (since H is symmetric) and satisfies the structural constraint (2e). Thus, to obtain a solution to the minimization problem (2), we perform an iterative procedure (starting from some random initial condition) that solves numerically the constraint equations (2b)-(2d) and the optimality equations (6a)-(6c). We then verify that these solutions satisfy the regularity and local minima equations (6d) and (6e), respectively. We repeat this procedure for several initial conditions to improve upon local solutions. However, due to the non-convexity of the minimization problem (2), convergence to a global minimum is not guaranteed. □

To conclude this section, we present a short numerical example that illustrates the conditions in Theorem 3.2.

Example 1: ***(Structured controllability radius of a line network)*** Consider a network with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

and input matrix $B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T}$. Notice that this is a line network controlled by the first node, which is known to be strongly structurally controllable [26]. We are interested in modifying only the existing edges of the line, that is, $\mathcal{H} = \mathcal{G}$.

With this sparsity constraint, the symmetric perturbation with the minimum Frobenius norm (global minimum) that renders the network uncontrollable is the one that removes the edge with the smallest weight [19]:

$$\Delta^*_{\text{global}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To verify our procedure against this result, we solve the constraint and optimality equations in Theorem 3.2. In addition to the above global minimum (with $v^*_{\text{global}} = [0, 0.7071, 0.7071]^\mathsf{T}$, $l^*_{\text{global}} = [5.6569, -1.7038, 1.7038]^\mathsf{T}$, $\lambda^*_{\text{global}} = 2$), we also obtain the following two local minima, where $\lambda^*_1 = \lambda^*_2 = 0$, and

$$\Delta^*_1, v^*_1, l^*_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8.1341 \\ 8 \\ 2.5228 \end{bmatrix}, \text{ and}$$

$$\Delta^*_2, v^*_2, l^*_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.9711 \\ -0.2387 \end{bmatrix}, \begin{bmatrix} 4.1191 \\ 5.1020 \\ 9.4922 \end{bmatrix},$$

which correspond to removing the edges between node 2 and 3, and all the edges of the network, respectively. □

IV. The Controllability Radius of Symmetric Brain and Random Networks

In the remainder part of the paper we focus on the numerical analysis and comparison of the controllability radius of brain and random networks. We focus on the case $\mathcal{H} = \mathcal{G}$, and consider the following network models.

Structural brain network (SBN). We use brain networks modeled by (1), where the anatomical connectivity matrices represent weighted adjacency matrices. To obtain the connectivity matrices, anatomical scans of 15 healthy subjects were parcellated according to the Lausanne atlas [8], and $n = 129$ regions are chosen as regions of interest. Figure 2 shows the main steps performed to obtain the connectivity matrices of structural brain networks. These network dynamics can be derived as a linearization of brain processes [27], and have been used, for instance, in [2], [6], [28]. Among these, [2] has numerically shown that this class of brain networks constructed from diffusion MRI data is controllable from one single node, which will support our assumption of selecting only one control node in our numerical study.

Random network (RN). Starting from a structural brain network, we generate a set of symmetric adjacency matrices by randomly permuting its edges, while maintaining connectivity and controllability from the selected control node.

In our study, we consider 15 SBN's and 10 RN's generated from each SBN, for a total of 165 networks. For each network, the solution to the optimization problem (2) is computed numerically. To do so, we run an extensive number of minimizations from random initial conditions. Finally, we compare the controllability radius of each SBN with the mean controllability radius of the 10 corresponding RN's.

We run two sets of numerical studies. First, we compare the structured controllability radius of brain and random networks for a fixed choice of control node, that is, the control node in a brain network and in all its random permutations is the same. Second, we compare brain and random networks after varying the control node to maximize the smallest eigenvalue of the controllability Gramian. The results of our studies are reported in Fig. 3.

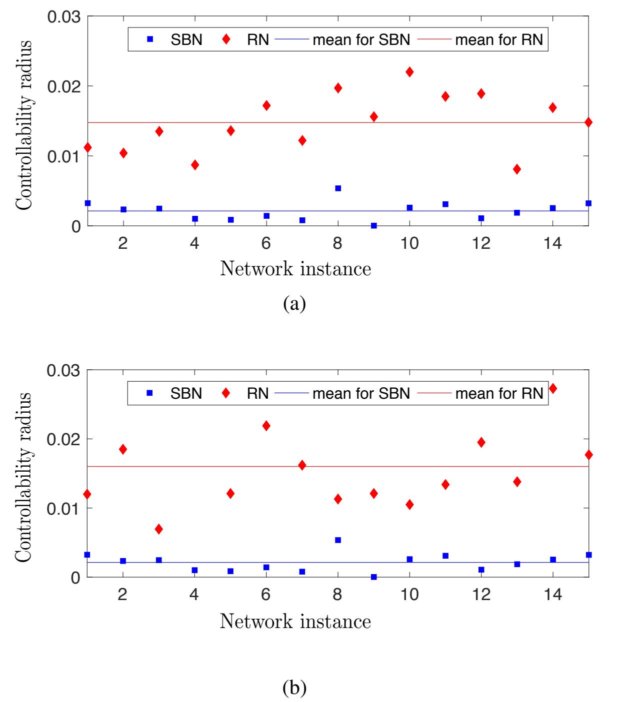

Fig. 3. Comparison between the controllability radii of 15 SBN's and 150 RN's. (a) The control node maximizes $\lambda_{\min}(W)$ for the i-th SBN, and the same control node is employed for the 10 randomized brain networks corresponding to the i-th SBN, $i = 1, \ldots, 15$. (b) The control node is selected differently in each network to maximize $\lambda_{\min}(W)$.

In both our sets of numerical studies, the results show that the controllability radius of structural brain networks is on average smaller than the controllability radius of the respective randomized versions. This result suggests that the topology of brain networks may not be accidental. Furthermore, the peculiar architecture of the brain could have evolved to favor dynamic features different from controllability by single regions [29]. This raises several questions, including characterizing the cost functions optimized by the anatomical structure of the brain [30].

The controllability radius of structural brain networks can be further exploited to provide interesting information on the effectiveness of network control from a certain area. In fact, it is possible to understand which areas make a better position for a control node without incurring in numerical artifacts that affect the spectral analysis of the controllability Gramian. For instance, when the brain network of subject 1 is controlled by the *right superiorparietal-3* region[3] (node 30), it displays the smallest controllability radius: $\|\Delta\|_F \approx 10^{-7}$. When the brain network of subject one is controlled by the *left isthmus of cingulate* (node 88), it displays the largest

[3]See [8] for the atlas with labels of the regions of interest in the brain.

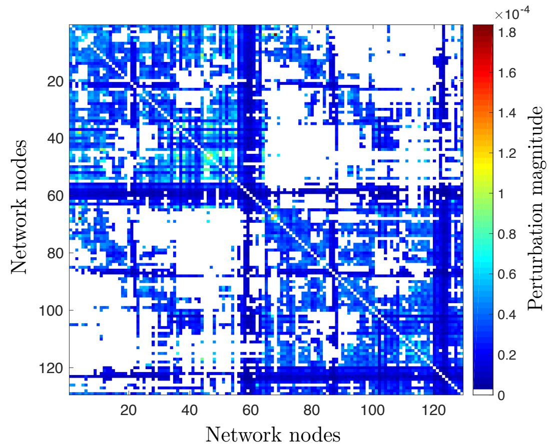

Fig. 4. Average magnitude of the weight changes among all the perturbations Δ_i for subject 1 when controlled by single brain regions, $i = 1, \ldots, n$.

controllability radius: $\|\Delta\|_F \approx 2.5 \cdot 10^{-2}$. This insight could inform the design of novel brain stimulation techniques.

To conclude, we noticed that the most perturbed interconnections correspond to entries with a small magnitude in the connectivity matrix. Furthermore, when comparing the mean perturbation for subject 1 (see Fig. 4) with the brain atlas, the areas where the most changes occur among all the perturbation matrices turn out to be the *frontal pole* (node 4 and 68) and the *pars orbitalis* (node 67). These two areas are indeed important for cortico-cortical control [2]. Assessing which interconnections tend to be more fragile toward uncontrollability of the brain and further enhancing the role that network controllability plays in the correct functioning of this complex organ opens new challenges for future research and may ultimately lead to the development of innovative personalized clinical therapies [1], [29].

V. Conclusion

In this paper we propose and analyze a novel notion of controllability for symmetric networks, namely the structured controllability radius. In particular, we quantify the controllability degree of a network with the smallest norm of a symmetric perturbation that renders the network uncontrollable and satisfies a pre-specified set of sparsity constraints. We derive a set of equations for the computation of the structured controllability radius, and illustrate our results through various examples including networks approximating a class of brain dynamics. Our numerical results show that brain networks feature a controllability radius that is consistently smaller than the one of random networks with similar weights, further highlighting the fundamental role of the organization of the brain for its dynamic functions.

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